

DONALDSON-THOMAS INVARIANTS AND FLOPS

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ABSTRACT. We prove a comparison formula for the Donaldson-Thomas curve-counting invariants of two smooth and projective Calabi-Yau threefolds related by a flop. By results of Bridgeland any two such varieties are derived equivalent. Furthermore there exist pairs of categories of perverse coherent sheaves on both sides which are swapped by this equivalence. Using the theory developed by Joyce we construct the motivic Hall algebras of these categories. These algebras provide a bridge relating the invariants on both sides of the flop.

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INTRODUCTION

In this paper we prove a comparison formula for the Donaldson-Thomas (DT) curve-counting invariants of two Calabi-Yau threefolds related by a flop. For us a Calabi-Yau threefold X will be a smooth and projective complex variety of dimension three with trivial canonical bundle and satisfying $H^1(X, \mathcal{O}_X) = 0$. For such an X we may write the generating series for the DT invariants

$$DT_X := \sum_{\beta, n} (-1)^n DT_X(\beta, n) q^{(\beta, n)}$$

where $\beta \in N_1(X)$ is the class of a curve in X , $n \in \mathbb{Z}$ is a zero-cycle and q is a formal variable. We can consider a flop of X , which is a second variety X^+ satisfying the same assumptions and fitting in a particular commutative diagram of birational morphisms

$$\begin{array}{ccc}
X & \xleftarrow{\phi} & X^+ \\
& \searrow f & \swarrow f^+ \\
& Y &
\end{array}$$

where f and f^+ contract curves in X and X^+ . We have the following result.

3.27 Theorem (continuing from p. 25). If we define the series¹

$$DT_f^\vee := \sum_{\substack{\beta, n \\ f_*\beta=0}} (-1)^n DT_X(-\beta, n) q^{(\beta, n)}$$

then the following identity is true

$$(\star) \quad DT_f^\vee \cdot DT_X = \phi_* \left(DT_{f^+}^\vee \cdot DT_{X^+} \right)$$

where ϕ_* is induced by ϕ .

This formula arises quite naturally using Bridgeland's derived equivalence [Bri02] between X^+ and X in combination with the motivic Hall algebra technology of Joyce and Song [JS08]. The importance of studying the case of flops lies in the fact that any birational map between Calabi-Yau threefolds can be decomposed into a sequence of flops. Let us succinctly explain how we deduce the formula.

We think of DT invariants as coming from a *weighted* Euler characteristic χ_B of the Hilbert scheme Hilb_X of X , where the weight is given by Behrend's microlocal function [Beh09]. Concretely, if $\text{Hilb}_X(\beta, n)$ is the scheme parameterising quotients of \mathcal{O}_X with Chern character $(0, 0, \beta, n)$ then $DT_X(\beta, n) = \chi_B(\text{Hilb}_X(\beta, n))$. By the work of Bridgeland we know that there exists a derived equivalence between X and X^+ . Furthermore we know that there exist abelian categories of *perverse coherent sheaves* inside the derived categories of X and X^+ which are swapped by this equivalence. The structure sheaf turns out to be a perverse coherent sheaf so we can speak of a *perverse Hilbert scheme* ${}^p\text{Hilb}_X$, parameterising perverse quotients of \mathcal{O}_X , and similarly for X^+ . We like to think of the weighted Euler characteristics of ${}^p\text{Hilb}_X$ as producing *relative* DT invariants and so we define $DT_{X/Y}(\beta, n) = \chi_B({}^p\text{Hilb}_X(\beta, n))$. Applying the derived equivalence we deduce the identity

$$DT_{X/Y} = \phi_* DT_{X^+/Y}.$$

Using techniques similar to the ones found in [Bri10a] we extract the comparison formula. What comes into play is the fact that the category of perverse coherent sheaves is a tilt, via a torsion pair, of the category of coherent sheaves. After constructing the motivic Hall algebra of perverse coherent sheaves, we can use the torsion pair to express $DT_{X/Y}^\vee$ in terms of DT_f^\vee and DT_X .

We should mention that Yukinobu Toda has given a different approach to the same problem [Tod09], using Van den Bergh's non-commutative resolution of Y [VdB04] and wall-crossing techniques. Our Theorem 3.27 is related to [Tod09, Theorem 5.8] via [Tod09, Theorem 5.6] (with slightly different notation). Strictly speaking, Toda's result applies to the naive counting invariants (defined using the ordinary Euler characteristic) and not to DT invariants, as the proof relies on a yet unproved (but expected) result regarding the local structure of the moduli stack of the objects of the derived category [Tod09, Conjecture 4.3].

Outline. In the first section we recall what we need about flops and construct the moduli stack of Bridgeland's perverse coherent sheaves. The second section is devoted to checking that Joyce's theory of motivic Hall algebras applies to perverse coherent sheaves. The third section contains the main result and its proof. We relegated to the appendix a few simple, but tedious, results about Lieblich's moduli stack of objects of the derived category [Lie06].

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¹This is the DT series of curves in X contracted by f and with a change of signs.

Conventions. In what follows \mathbb{C} will denote the field of complex numbers and all stacks and morphisms will be over \mathbb{C} . Given a scheme (X, \mathcal{O}_X) we denote by $D(\mathcal{O}_X)$ the derived category of \mathcal{O}_X -modules, by $D(X) = D_{\text{coh}}(\mathcal{O}_X)$ the derived category of \mathcal{O}_X -modules with coherent cohomology and by $D^b(X)$ the corresponding bounded version. Given a complex $E \in D(\mathcal{O}_X)$ we denote by $H^i(E) \in \mathcal{O}_X\text{-Mod}$ the i -th cohomology sheaf and by $H^i(X, E) = R^i\Gamma(X, E)$ the i -th (hyper)cohomology group. Whenever we have a diagram of schemes $T \xrightarrow{u} S \xleftarrow{\pi} X$ we often denote a fibre product as X_T together with induced maps $\pi_T : X_T \rightarrow T$, $u_X : X_T \rightarrow X$. The derived pullback $Lu_X^* E$ of an object $E \in D(\mathcal{O}_X)$ will simply be denoted by $E|_{X_T}^L$. All schemes (and all algebraic stacks) will be assumed to be locally of finite type over \mathbb{C} .

1. FLOPS

In this section we recall a few facts about the categories of perverse coherent sheaves and we construct the corresponding moduli spaces.

1.1. Perverse Coherent Sheaves. Henceforth we assume to be working within the following setup.

Situation 1.1

Fix a smooth and projective variety X , over \mathbb{C} , with trivial canonical bundle $\omega_X \cong \mathcal{O}_X$ and satisfying $H^1(X, \mathcal{O}_X) = 0$. Fix a map $f : X \rightarrow Y$ satisfying the following properties:

- the morphism f is proper, birational and an isomorphism in codimension one;
- Y is projective and Gorenstein;
- the dualising sheaf of Y is trivial: $\omega_Y \cong \mathcal{O}_Y$;
- $Rf_* \mathcal{O}_X = \mathcal{O}_Y$.

Fix a commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{\phi} & X^+ \\ & \searrow f & \swarrow f^+ \\ & Y & \end{array}$$

where the pair (X^+, f^+) satisfies the same assumptions as (X, f) above and ϕ is birational but not an isomorphism.

Notation. We loosely refer to such an f as a *flopping contraction* and to X^+ as the *flop* of X (along f). We denote by $\mathcal{A} = \text{Coh } X$ the category of coherent sheaves on X and similarly $\mathcal{A}^+ = \text{Coh } X^+$. We write ${}^p\mathcal{A}, {}^p\mathcal{A}^+$ ($p = -1, 0$) for the categories of perverse coherent sheaves on X and X^+ (defined below).

Notice that for any coherent sheaf E on X , $R^i f_* E = 0$ for $i \geq 2$.

An interesting fact proved by Bridgeland [Bri02] is that we can characterise the flop X^+ as a moduli space of “point-like objects” of a suitable abelian category

$${}^p\mathcal{A} = {}^p\text{Coh}(X/Y) \subset D^b(X)$$

called the category of *perverse coherent sheaves*. There are actually two versions of this category, indexed by a choice of superscript $p = -1, 0$. We recall their definition:

$$(1.2) \quad {}^p\mathcal{A} = \left\{ E \in D^b(X) \mid Rf_* E \in \text{Coh } Y, \text{Ext}_X^{<-p}(C, E) = \text{Ext}_X^{<-p}(E, C) = 0 \right\}$$

where $p = -1, 0$ and

$$\mathcal{C} = \{ E \in \mathcal{A} \mid Rf_* E = 0 \}$$

where $\mathcal{A} = \text{Coh } X$. These categories can actually be characterised as tilts of appropriate torsion pairs [BR07] on \mathcal{A} as follows [VdB04, Section 3]. Consider the following subcategories of \mathcal{A} :

$$\begin{aligned} {}^0\mathcal{T} &= \{T \in \mathcal{A} \mid R^1 f_* T = 0\} \\ {}^{-1}\mathcal{T} &= \{T \in \mathcal{A} \mid R^1 f_* T = 0, \text{Hom}(T, \mathcal{C}) = 0\} \\ {}^{-1}\mathcal{F} &= \{F \in \mathcal{A} \mid f_* F = 0\} \\ {}^0\mathcal{F} &= \{F \in \mathcal{A} \mid f_* F = 0, \text{Hom}(\mathcal{C}, F) = 0\}. \end{aligned}$$

We have that $({}^p\mathcal{T}, {}^p\mathcal{F})$ forms a torsion pair in \mathcal{A} ($p = -1, 0$) and the tilt of \mathcal{A} with respect to it coincides with the category ${}^p\mathcal{A}$ above. Notice that we picked the convention where

$${}^p\mathcal{F}[1] \subset {}^p\mathcal{A} \subset D^{[-1, 0]}(X).$$

We remark in passing that $\mathcal{O}_X \in {}^p\mathcal{T}$.

Moreover Bridgeland [Bri02] showed that there exists a derived equivalence Φ between X^+ and X which sends ${}^q\mathcal{A}^+$ to ${}^p\mathcal{A}$, where $q = -(p + 1)$. Before moving on we state an easy lemma.

1.3 Lemma. For all $T \in {}^p\mathcal{T}$ we have $H^i(X, T) = H^i(Y, f_* T)$, for all i . For all $F \in {}^p\mathcal{F}$ we have $H^1(X, F) = H^0(Y, R^1 f_* F)$ and $H^i(X, F) = 0$ for all $i \neq 1$.

For a proof one may use Leray's spectral sequence and the fact that sheaves in ${}^p\mathcal{F}$ are supported in dimension one.

1.2. Moduli. To define the motivic Hall algebra of ${}^p\mathcal{A}$ in the next section we need, first of all, an algebraic stack ${}^p\mathfrak{A}$ parameterising objects of ${}^p\mathcal{A}$. We build it as a substack of the stack \mathfrak{Mum}_X , which was constructed by Lieblich [Lie06] and named *the mother of all moduli of sheaves*. For its definition and some further properties we refer the reader to the appendix. We only recall that \mathfrak{Mum}_X parameterises objects in the derived category of X with no negative self-extensions. This last condition is key to avoid having to enter the realm of higher stacks. We remark that as ${}^p\mathcal{A}$ is the heart of a t-structure its objects satisfy this condition. In addition to simplifying matters technically, this approach allows to view both ${}^p\mathfrak{A}$ and \mathfrak{A} (the second being the stack of coherent sheaves on X) as sitting inside the big stack \mathfrak{Mum}_X , which will be useful later to compare the Behrend function of a substack of both \mathfrak{A} and ${}^p\mathfrak{A}$ such as ${}^p\mathfrak{T}$, the stack parameterising objects in ${}^p\mathcal{T}$.

Notice that the definition of ${}^p\mathcal{A}$ is independent of the ground field. Concretely, take $E \in \mathfrak{Mum}_X(T)$ a family of complexes over X parameterised by T and $t : \text{Spec } k \rightarrow T$ a geometric point. We can consider $E|_{X_t}^L$, the derived restriction of E to the fibre X_t of X_T over t , and it makes sense to write $E|_{X_t}^L \in {}^p\mathcal{A}$ (where the latter category is interpreted relatively to k).

1.4 Proposition. Define a prestack² by the rule

$${}^p\mathfrak{A}(T) = \{E \in \mathfrak{Mum}_X(T) \mid \forall t \in T, E|_{X_t}^L \in {}^p\mathcal{A}\}$$

with restriction maps induced by \mathfrak{Mum}_X and where by $t \in T$ we mean that $t : \text{Spec } k \rightarrow T$ is a geometric point of T . The prestack ${}^p\mathfrak{A}$ is an open substack of \mathfrak{Mum}_X .

Proof: Firstly, the condition $Rf_{t,*} E|_{X_t}^L \in \text{Coh } Y_t$ is open. This follows from the proof of Proposition A.8.

Secondly, it is sufficient to check the condition

$$\text{Ext}_{X_t}^{<-p}(\mathcal{C}, E|_{X_t}^L) = 0$$

for only a finite number of elements of \mathcal{C} . This follows from [VdB04, Section 3]. The dual statement (for $\text{Ext}_X^{<-p}(-, \mathcal{C})$) holds for analogous reasons.

Thirdly, the conditions $\text{Ext}_{X_t}^{<-p}(G|_{X_t}^L, E|_{X_t}^L) = 0$ and $\text{Ext}_{X_t}^{<-p}(E|_{X_t}^L, G|_{X_t}^L)$, for $G \in \text{Coh } X$, are open. This follows from base change [Lip09, Theorem 3.10.3] and compatibility between $R\text{Hom}$ and derived pullback. Explicitly we are asking whether the set of points t such that $R\pi_{t,*} R\text{Hom}_{X_t}(E|_{X_t}^L, G|_{X_t}^L) \in D^{<-p}(\text{Spec } k)$ is open. As

$$R\text{Hom}_{X_t}(E|_{X_t}^L, G|_{X_t}^L) \simeq R\text{Hom}_{X_T}(E, G|_{X_T}^L)|_{X_t}^L$$

²We use the term *prestack* in analogy with *presheaf*.

we already know this set to be open (the proof for the dual statement is analogous). \blacksquare

It will be important for us to also have moduli spaces for the torsion and torsion-free subcategories ${}^p\mathcal{T}$, ${}^p\mathcal{F}$. We define them similarly as above.

$$\begin{aligned} {}^p\mathfrak{F}(\mathcal{T}) &= \{E \in \mathfrak{A}_X(\mathcal{T}) \mid \forall t \in \mathcal{T}, E|_{X_t}^L \in {}^p\mathcal{F}\} \\ {}^p\mathfrak{T}(\mathcal{T}) &= \{E \in \mathfrak{A}_X(\mathcal{T}) \mid \forall t \in \mathcal{T}, E|_{X_t}^L \in {}^p\mathcal{T}\} \end{aligned}$$

Arguing similarly as in the proof above one has the expected open inclusions of algebraic stacks

$$\begin{aligned} {}^p\mathfrak{T}, {}^p\mathfrak{F} &\subset \mathfrak{A} \subset \mathfrak{Mum}_X^{[-1,0]} \\ {}^p\mathfrak{T}, {}^p\mathfrak{F}[1] &\subset {}^p\mathfrak{A} \subset \mathfrak{Mum}_X^{[-1,0]} \end{aligned}$$

where $\mathfrak{Mum}_X^{[-1,0]}$ is the substack of \mathfrak{Mum}_X parameterising complexes concentrated in degrees -1 and 0 .

We conclude this section with a result regarding the structure of ${}^p\mathfrak{A}$. This result essentially allows us to carry all the proofs to set up the motivic Hall algebra of ${}^p\mathcal{A}$ from the case of coherent sheaves. Let \mathfrak{A}^+ denote the stack of coherent sheaves on X^+ .

1.5 Proposition. There is a collection of open substacks ${}^p\mathfrak{A}_n \subset {}^p\mathfrak{A}$ which jointly cover ${}^p\mathfrak{A}$. Each ${}^p\mathfrak{A}_n$ is isomorphic to an open substack of \mathfrak{A} , for $p = -1$, and to an open substack of \mathfrak{A}^+ , for $p = 0$.

To prove this result we start by remarking that, as a consequence of our assumptions on X , the structure sheaf \mathcal{O}_X is a spherical object [Huy06, Definition 8.1] in $D^b(X)$. Thus the Seidel-Thomas spherical twist around it is an autoequivalence of $D^b(X)$. This functor can be explicitly described as the Fourier-Mukai transform with kernel the ideal sheaf of the diagonal of X shifted by one. We thus get an exact autoequivalence τ of $D^b(X)$ and we notice that the subcategory of complexes with no negative self-extensions is invariant under τ . As Fourier-Mukai transforms behave well in families [BBHR09, Proposition 6.1] we also obtain an isomorphism (which by abuse of notation we still denote by τ) of the stack \mathfrak{Mum}_X .

Let us now fix an ample line bundle L on the base of the flopping contraction Y . Tensoring with f^*L^n also induces an automorphism of \mathfrak{Mum}_X . The automorphism $\tau_n \in \text{Aut}(\mathfrak{Mum}_X)$ is then defined by $\tau_n(E) = \tau(E \otimes f^*L^n)$. The following lemma tells us how to use the automorphisms τ_n to deduce the proposition above.

1.6 Lemma. Let $p = -1$ and let $E \in {}^p\mathcal{A}$ be a perverse coherent sheaf. Then there exists an n_0 such that for all $n \geq n_0$

$$\widetilde{\tau}_n(E) = \tau_n(E)[-1] \in \mathcal{A}.$$

Proof: As in the statement, we fix $p = -1$. The two key properties we use of τ_n are that it is an exact functor and that for a complex G we have an exact triangle

$$H^\bullet(X, G(n)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\text{ev}} G(n) \rightarrow \tau_n(G) \rightarrow$$

where $G(n) = G \otimes_{\mathcal{O}_X} f^*L^n$.

Let now $E \in {}^p\mathcal{A}$ be a perverse coherent sheaf together with its torsion pair exact sequence (in ${}^p\mathcal{A}$)

$$F[1] \hookrightarrow E \rightarrow T$$

where $F \in {}^p\mathcal{F}$, $T \in {}^p\mathcal{T}$. Using Leray's spectral sequence, Lemma 1.3 and Serre vanishing on Y we can pick n big enough so that all hypercohomologies involved, $H^\bullet(X, F(n))$, $H^\bullet(X, T(n))$, $H^\bullet(X, E(n))$, are concentrated in degree zero.

From the triangle

$$H^\bullet(X, E(n)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\text{ev}} E(n) \rightarrow \tau_n(E) \rightarrow$$

we have that $\tau_n(E) \in D^{[-1,0]}$, similarly for $\tau_n(F[1])$ and $\tau_n(T)$. From the triangle

$$H^\bullet(X, F[1](n)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\text{ev}} F[1](n) \rightarrow \tau_n(F[1]) \rightarrow$$

we obtain that $H^0(\tau_n(F[1])) = 0$.

From the triangle

$$\tau_n(F[1]) \rightarrow \tau_n(E) \rightarrow \tau_n(T) \rightarrow$$

arising from exactness of τ_n we have that $H^0(\tau_n(T)) \simeq H^0(\tau_n(E))$. Thus to prove the lemma it suffices to show that $H^0(\tau_n(T)) = 0$.

Finally, from the triangle

$$H^\bullet(X, T(n)) \otimes_{\mathbb{C}} \mathcal{O}_X \rightarrow T(n) \rightarrow \tau_n(T) \rightarrow$$

one obtains the following exact sequence.

$$0 \rightarrow H^{-1}(\tau_n(T)) \rightarrow H^0(X, T(n)) \otimes_{\mathbb{C}} \mathcal{O}_X \xrightarrow{\alpha} T(n) \xrightarrow{\beta} H^0(\tau_n(T)) \rightarrow 0$$

Thus we have

$$\tau_n(E)[-1] \in \mathcal{A} \iff H^0(\tau_n(E)) \simeq H^0(\tau_n(T)) = 0 \iff \beta = 0.$$

Let $K = \ker \beta$. We then have two short exact sequences

$$\begin{array}{ccccc} H^{-1}(\tau_n(T)) & \hookrightarrow & H^0(X, T(n)) \otimes_{\mathbb{C}} \mathcal{O}_X & \xrightarrow{\gamma} & K \\ K & \xhookrightarrow{\delta} & T(n) & \xrightarrow{\beta} & H^0(\tau_n(T)) \end{array}$$

and notice that $\delta\gamma = \alpha$. By pushing forward the first sequence via f_* we have that $R^1 f_* K = 0$, as $R^1 f_* \mathcal{O}_X = 0$. Pushing forward the second sequence yields the exact sequence

$$f_* K \hookrightarrow f_* T(n) \twoheadrightarrow f_* H^0(\tau_n(T))$$

and $R^1 f_* H^0(\tau_n(T)) = 0$, as $R^1 f_* T(n) = 0$ (this last is a consequence of Lemma 1.3 and the projection formula).

By taking n even bigger we can assume $f_* T(n)$ to be generated by global sections and thus we can assume $f_* \alpha$ to be surjective. As $\alpha = \delta\gamma$ we obtain that $f_* \delta$ is surjective and thus $f_* H^0(\tau_n(T)) = 0$. As a consequence we have that $H^0(\tau_n(T)) \in \mathcal{C}$.

The sheaf $T(n)$ is in ${}^p\mathcal{T}$ (this is a simple computation, the key fact to notice is that $\mathcal{C}(n) = \mathcal{C}$). Finally, as $T(n) \in {}^p\mathcal{T}$ and $H^0(\tau_n(T)) \in \mathcal{C}$, $\beta = 0$. \blacksquare

To prove Proposition 1.5 we define ${}^p\mathcal{A}_n$ to be the subcategory of ${}^p\mathcal{A}$ consisting of elements E such that $\tau_n(E) \in \mathcal{A}$. We can produce a moduli stack for ${}^p\mathcal{A}_n$ via the following composition of cartesian diagrams.

$$\begin{array}{ccccc} {}^p\mathcal{A}_n & \hookrightarrow & \widetilde{\tau}_n^{-1}(\mathcal{A}) & \longrightarrow & \mathcal{A} \\ \downarrow & & \downarrow & & \downarrow \\ {}^p\mathcal{A} & \hookrightarrow & \mathfrak{Mum}_X & \xrightarrow{\widetilde{\tau}_n} & \mathfrak{Mum}_X \end{array}$$

We obtain that ${}^p\mathcal{A}_n$ is an open substack of ${}^p\mathcal{A}$ and is isomorphic to an open substack of \mathcal{A} via $\widetilde{\tau}_n$. From the previous lemma we have that the sum of the inclusions $\coprod_n {}^p\mathcal{A}_n \rightarrow {}^p\mathcal{A}$ is surjective. For $p = 0$ we use the fact that Bridgeland's derived equivalence Φ is a Fourier-Mukai transform and thus induces an isomorphism of stacks $\mathfrak{Mum}_{X^+} \rightarrow \mathfrak{Mum}_X$, which takes ${}^0\mathcal{A}^+$ to ${}^0\mathcal{A}$.

2. HALL ALGEBRAS

This section is devoted to constructing the motivic Hall algebra of perverse coherent sheaves. We start by recalling the general setup and then move on to check that we can port the construction of the Hall algebra of coherent sheaves to the perverse case using Proposition 1.5.

2.1. Grothendieck Rings and the Hall Algebra of Coherent Sheaves. In this section we construct the Hall algebra $H({}^p\mathcal{A})$ of our perverse coherent sheaves, which is a module over $K(\text{St}/\mathbb{C})$, the Grothendieck ring of stacks over \mathbb{C} . We start by recalling the definition of the latter. All the omitted proofs can be found, for example, in [Bri10b].

2.1 Definition. The *Grothendieck ring of schemes* $K(\text{Sch}/\mathbb{C})$ is defined to be the \mathbb{Q} -vector space spanned by isomorphism classes of schemes of finite type over \mathbb{C} modulo the *cut* & *paste* relations:

$$[X] = [Y] + [X \setminus Y]$$

for all Y closed in X . The ring structure is induced by $[X \times Y] = [X] \cdot [Y]$.

Notice that the zero element is given by the empty scheme and the unit for the multiplication is given by $[\text{Spec } \mathbb{C}]$. Also, the Grothendieck ring disregards any non-reduced structure, as $[X_{\text{red}}] = [X] - 0$. This ring can equivalently be described in terms of geometric bijections and Zariski fibrations.

2.2 Definition. A morphism $f : X \rightarrow Y$ of finite type schemes is a *geometric bijection* if it induces a bijection on \mathbb{C} -points $f(\mathbb{C}) : X(\mathbb{C}) \rightarrow Y(\mathbb{C})$.

A morphism $p : X \rightarrow Y$ is a *Zariski fibration* if there exists a *trivialising* Zariski open cover of Y . That is, there exists a Zariski open cover $\{Y_i\}_i$ of Y together with schemes F_i such that $p^{-1}(Y_i) \cong Y_i \times F_i$, as Y_i -schemes.

Two Zariski fibrations $p : X \rightarrow Y$, $p' : X' \rightarrow Y$ *have the same fibres* if there exists a trivialising open cover for both fibrations such that the fibres are isomorphic $F_i \cong F'_i$.

2.3 Lemma. We can describe the ring $K(\text{Sch}/\mathbb{C})$ as the \mathbb{Q} -vector space spanned by isomorphism classes of schemes of finite type over \mathbb{C} modulo the following relations.³

- (1) $[X_1 \sqcup X_2] = [X_1] + [X_2]$, for every pair of schemes X_1, X_2 .
- (2) $[X_1] = [X_2]$ for every geometric bijection $f : X_1 \rightarrow X_2$.
- (3) $[X_1] = [X_2]$ for every pair of Zariski fibrations $p_i : X_i \rightarrow Y$ with same fibres.

We now consider the Grothendieck ring of stacks.

2.4 Definition. A morphism of finite type algebraic stacks $f : X_1 \rightarrow X_2$ is a *geometric bijection* if it induces an equivalence of groupoids on \mathbb{C} -points $f(\mathbb{C}) : X_1(\mathbb{C}) \rightarrow X_2(\mathbb{C})$.⁴

A morphism of algebraic stacks $p : X \rightarrow Y$ is a *Zariski fibration* if given any morphism from a scheme $T \rightarrow Y$ the induced map $X \times_Y T \rightarrow T$ is a Zariski fibration of schemes. In particular a Zariski fibration is a schematic morphism.

Two Zariski fibrations between algebraic stacks $p_i : X_i \rightarrow Y$ *have the same fibres* if the two maps $X_i \times_Y T \rightarrow T$ induced by a morphism from a scheme $T \rightarrow Y$ are two Zariski fibrations with the same fibres.

2.5 Definition. The *Grothendieck ring of stacks* $K(\text{St}/\mathbb{C})$ is defined to be the \mathbb{Q} -vector space spanned by isomorphism classes of Artin stacks of finite type over \mathbb{C} with affine geometric stabilisers, modulo the following relations.

- (1) $[X_1 \sqcup X_2] = [X_1] + [X_2]$ for every pair of stacks X_1, X_2 .
- (2) $[X_1] = [X_2]$ for every geometric bijection $f : X_1 \rightarrow X_2$.
- (3) $[X_1] = [X_2]$ for every pair of Zariski fibrations $p_i : X_i \rightarrow Y$ with the same fibres.

Let us call $\mathbb{L} = [\mathbb{A}^1]$ the element represented by the affine line. There obvious ring homomorphism $K(\text{Sch}/\mathbb{C}) \rightarrow K(\text{St}/\mathbb{C})$ becomes an isomorphism after inverting the elements \mathbb{L} and $(\mathbb{L}^k - 1)$, for $k \geq 1$ [Bri10b, Lemma 3.9]. Thus the ring homomorphism factors as follows.

$$K(\text{Sch}/\mathbb{C}) \rightarrow K(\text{Sch}/\mathbb{C})[\mathbb{L}^{-1}] \rightarrow K(\text{St}/\mathbb{C})$$

We also mention that through the lens of the Grothendieck ring one cannot tell apart varieties from schemes or even algebraic spaces [Bri10b, Lemma 2.12].

It also makes sense to speak of a *relative* Grothendieck group $K(\text{St}/S)$, where S is a fixed base stack which we assume to be Artin, *locally* of finite type over \mathbb{C} and with affine geometric stabilisers. We define $K(\text{St}/S)$ to be spanned by isomorphism classes of morphisms $[W \rightarrow S]$ where W is an Artin stack of finite type over \mathbb{C} with affine geometric stabilisers, modulo the following relations.

- (1) $[f_1 \sqcup f_2 : X_1 \sqcup X_2 \rightarrow S] = [X_1 \xrightarrow{f_1} S] + [X_2 \xrightarrow{f_2} S]$, for every pair of stacks X_i .
- (2) For a morphism $f : X_1 \rightarrow X_2$ over S , with f a geometric bijection, $[X_1 \rightarrow S] = [X_2 \rightarrow S]$.
- (3) For every pair of Zariski fibrations with the same fibres $X_1 \rightarrow Y \leftarrow X_2$ and every morphism $Y \rightarrow S$

$$[X_1 \rightarrow Y \rightarrow S] = [X_2 \rightarrow Y \rightarrow S].$$

Given a morphism $a : S \rightarrow T$ we have a pushforward map

$$\begin{aligned} a_* : K(\text{St}/S) &\longrightarrow K(\text{St}/T) \\ [X \rightarrow S] &\longmapsto [X \rightarrow S \xrightarrow{a} T] \end{aligned}$$

³The three relations we present here are actually redundant, cf. [Bri10b, Lemma 2.9], although the same is not true for stacks.

⁴We point out that geometric bijections are representable by algebraic spaces [AH11, Lemma 2.3.9].

and given a morphism of finite type $b : S \rightarrow T$ we have a pullback map

$$\begin{aligned} b^* : K(\mathrm{St}/T) &\longrightarrow K(\mathrm{St}/S) \\ [X \rightarrow T] &\longmapsto [X \times_T S \rightarrow S]. \end{aligned}$$

The pushforward and pullback are functorial and satisfy base-change. Furthermore, given a pair of stacks S_1, S_2 there is a Künneth map

$$\begin{aligned} \kappa : K(\mathrm{St}/S_1) \otimes K(\mathrm{St}/S_2) &\longrightarrow K(\mathrm{St}/S_1 \times S_2) \\ [X_1 \rightarrow S_1] \otimes [X_2 \rightarrow S_2] &\longmapsto [X_1 \times X_2 \rightarrow S_1 \times S_2]. \end{aligned}$$

Take now \mathfrak{A} to be the stack of coherent sheaves on X , where X is smooth and projective over \mathbb{C} , and denote by $H(\mathcal{A})$ the Grothendieck ring $K(\mathrm{St}/\mathfrak{A})$ (where \mathcal{A} stands for $\mathrm{Coh} X$). We can endow $H(\mathcal{A})$ with a *convolution product*, coming from the abelian structure of \mathcal{A} . The product is defined as follows. Let $\mathfrak{A}^{(2)}$ be the stack of exact sequences in \mathcal{A} . There are three natural morphisms $a_1, b, a_2 : \mathfrak{A}^{(2)} \rightarrow \mathfrak{A}$ which take an exact sequence

$$A_1 \hookrightarrow B \twoheadrightarrow A_2$$

to A_1, B, A_2 respectively. Consider the following diagram.

$$\begin{array}{ccc} \mathfrak{A}^{(2)} & \xrightarrow{b} & \mathfrak{A} \\ (a_1, a_2) \downarrow & & \\ \mathfrak{A} \times \mathfrak{A} & & \end{array}$$

We remark that (a_1, a_2) is of finite type [Bri10b, Lemma 4.2]. A *convolution product* can be then defined as follows:

$$\begin{aligned} m : H(\mathcal{A}) \otimes H(\mathcal{A}) &\longrightarrow H(\mathcal{A}) \\ m &= b_*(a_1, a_2)^* \kappa. \end{aligned}$$

Explicitly, given two elements $[X_1 \xrightarrow{f_1} \mathfrak{A}]$, $[X_2 \xrightarrow{f_2} \mathfrak{A}]$ we write $f_1 * f_2 = m(f_1 \otimes f_2)$ for their product which is given by the top row of the following diagram.

$$\begin{array}{ccccc} & & f_1 * f_2 & & \\ & \text{---} \text{dashed arc} \text{---} & & & \\ Z & \xrightarrow{\quad} & \mathfrak{A}^{(2)} & \xrightarrow{b} & \mathfrak{A} \\ \downarrow & \square & \downarrow (a_1, a_2) & & \\ X_1 \times X_2 & \xrightarrow{f_1 \times f_2} & \mathfrak{A} \times \mathfrak{A} & & \end{array}$$

The convolution product endows $H(\mathcal{A})$ with an associative $K(\mathrm{St}/\mathbb{C})$ -algebra structure with unit element given by $[\mathrm{Spec} \mathbb{C} = \mathfrak{A}_0 \subset \mathfrak{A}]$, the inclusion of the zero object.

2.2. The Hall Algebra of Perverse Coherent Sheaves. We now assume to be working in Situation 1.1. We want to replace \mathcal{A} by ${}^p\mathcal{A}$ and construct the analogous algebra $H({}^p\mathcal{A})$. We first need the moduli stack ${}^p\mathfrak{A}^{(2)}$ parameterising short exact sequences in ${}^p\mathcal{A}$. Define a prestack ${}^p\mathfrak{A}^{(2)}$ by assigning to each scheme T the groupoid ${}^p\mathfrak{A}^{(2)}(T)$ whose objects are exact triangles

$$E_1 \rightarrow E \rightarrow E_2 \twoheadrightarrow$$

with vertices belonging to ${}^p\mathfrak{A}(T)$ and whose morphisms are isomorphisms of triangles. The restriction functors are given by derived pullback, which is an exact functor so takes exact triangles to exact triangles.

2.6 Proposition. The prestack ${}^p\mathfrak{A}^{(2)}$ is an Artin stack locally of finite type over \mathbb{C} with affine stabilisers.

Proof: This prestack is well-defined and satisfies descent as $\mathrm{Ext}_{p\mathcal{A}}^{\leq 0}(A, B)$ vanishes for any two objects $A, B \in {}^p\mathcal{A}$. Take now $p = -1$. We want to use the functors $\tilde{\tau}_n$ of Lemma 1.6. Notice that the subcategory ${}^p\mathcal{A}_n \subset {}^p\mathcal{A}$, of objects which become coherent after a twist by $\tilde{\tau}_n$, is extension-closed and hence we have a well-defined stack of exact sequences ${}^p\mathfrak{A}_n^{(2)}$, which is an open substack of ${}^p\mathfrak{A}^{(2)}$. Using Proposition 1.5 and the fact that $\tilde{\tau}_n$ is an exact functor we can embed ${}^p\mathfrak{A}_n^{(2)}$ inside $\mathfrak{A}^{(2)}$, thus proving that ${}^p\mathfrak{A}_n^{(2)}$ is algebraic.

The sum $\coprod_n {}^p\mathfrak{A}_n^{(2)} \rightarrow {}^p\mathfrak{A}^{(2)}$ is surjective and thus the stack ${}^p\mathfrak{A}^{(2)}$ is algebraic. All other properties are deduced by the fact that ${}^p\mathfrak{A}_n^{(2)}$ is an open substack of $\mathfrak{A}^{(2)}$. The case $p = 0$ is dealt with by passing to the flop X^+ . ■

The proof actually produces more: it gives an analogue to Proposition 1.5.

As for coherent sheaves, the stack ${}^p\mathfrak{A}^{(2)}$ comes equipped with three morphisms a_1, b, a_2 , sending a triangle of perverse coherent sheaves

$$E_1 \rightarrow E \rightarrow E_2 \rightrightarrows$$

to E_1, E, E_2 respectively. The exact functor $\tilde{\tau}_n$ yields a commutative diagram ($p = -1$)

$$\begin{array}{ccc} {}^p\mathfrak{A}_n^{(2)} & \hookrightarrow & \mathfrak{A}^{(2)} \\ (a_1, a_2) \downarrow & & \downarrow \\ {}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n & \hookrightarrow & \mathfrak{A} \times \mathfrak{A} \end{array}$$

where the vertical arrow on the right is the corresponding morphism for coherent sheaves, which is of finite type. From this last observation and the fact that being of finite type is local on the target, we automatically have that the (global) morphism $(a_1, a_2) : {}^p\mathfrak{A}^{(2)} \rightarrow {}^p\mathfrak{A}^2$ is of finite type. To define the convolution product on $K(\mathrm{St}/{}^p\mathfrak{A})$ (or equivalently the algebra structure of $H({}^p\mathcal{A})$) we may proceed analogously as for coherent sheaves. Once again, to deal with $p = 0$ one passes over to the flop X^+ .

2.3. More Structure on Hall Algebras. There is a natural way to bestow a grading upon our Hall algebras. Recall that the Grothendieck group $K(D^b(X))$ can be viewed as both $K(\mathcal{A})$ or $K({}^p\mathcal{A})$. The *Euler form* χ is defined as

$$\chi(E, F) = \sum_j (-1)^j \dim_{\mathbb{C}} \mathrm{Ext}_X^j(E, F)$$

on coherent sheaves E, F and then extended to the whole of $K(X)$. By Serre duality the left and right radicals of χ are equal and we define the *numerical* Grothendieck group of X as $N(X) = K(X)/K(X)^\perp$. As the numerical class of a complex stays constant in families we have a decomposition

$$\mathfrak{Mum}_X = \bigsqcup_{\alpha \in N(X)} \mathfrak{Mum}_{X, \alpha}$$

where $\mathfrak{Mum}_{X, \alpha}$ parameterises complexes of class α . Let Γ denote the *positive cone* of coherent sheaves, i.e. the image of objects of \mathcal{A} inside $N(X)$. It is a submonoid of $N(X)$ and for \mathfrak{A} the previous decomposition can be refined to

$$\mathfrak{A} = \bigsqcup_{\alpha \in \Gamma} \mathfrak{A}_\alpha.$$

We can also define sub-modules $H(\mathcal{A})_\alpha \subset H(\mathcal{A})$, where $H(\mathcal{A})_\alpha$ denotes $K(\mathrm{St}/\mathfrak{A}_\alpha)$ (which can be thought as spanned by classes of morphisms $[W \rightarrow \mathfrak{A}]$ factoring through \mathfrak{A}_α). We then get a Γ -grading

$$H(\mathcal{A}) = \bigoplus_{\alpha \in \Gamma} H(\mathcal{A})_\alpha.$$

Analogously, we have a positive cone ${}^p\Gamma \subset N(X)$ of perverse coherent sheaves. The Hall algebra thus decomposes as

$$H({}^p\mathcal{A}) = \bigoplus_{\alpha \in {}^p\Gamma} H({}^p\mathcal{A})_\alpha.$$

We mentioned earlier that the morphism from the Grothendieck ring of varieties to the Grothendieck ring of stacks factors as follows

$$K(\text{Sch}/\mathbb{C}) \rightarrow K(\text{Sch}/\mathbb{C})[\mathbb{L}^{-1}] \rightarrow K(\text{St}/\mathbb{C}).$$

Let $R = K(\text{Sch}/\mathbb{C})[\mathbb{L}^{-1}]$. One can define a subalgebra [Bri10b, Theorem 5.1] $H_{\text{reg}}(\mathcal{A})$ of *regular elements* as the R -module spanned by classes $[W \rightarrow \mathfrak{A}]$ with W a scheme. We have an analogous setup for perverse coherent sheaves.

2.7 Proposition. Let $H_{\text{reg}}({}^p\mathcal{A})$ to be the sub- R -module spanned by classes $[W \rightarrow {}^p\mathfrak{A}]$ with W a scheme. Then $H_{\text{reg}}({}^p\mathcal{A})$ is closed under the convolution product and the quotient

$$H_{\text{sc}}({}^p\mathcal{A}) = H_{\text{reg}}({}^p\mathcal{A})/(\mathbb{L} - 1)H_{\text{reg}}({}^p\mathcal{A})$$

is a commutative $K(\text{Sch}/\mathbb{C})$ -algebra.

Proof: Once again, we may appeal to the case of coherent sheaves by using the functors $\widetilde{\tau}_n$. Assume $p = -1$ and let $[f_1 : S_1 \rightarrow {}^p\mathfrak{A}]$, $[f_2 : S_2 \rightarrow {}^p\mathfrak{A}]$ be two elements of $H({}^p\mathcal{A})$ such that the S_i are schemes. Consider the two morphisms

$$\begin{aligned} f_1 \times f_2 : S_1 \times S_2 &\rightarrow {}^p\mathfrak{A} \times {}^p\mathfrak{A} \\ (a_1, a_2) : {}^p\mathfrak{A}^{(2)} &\rightarrow {}^p\mathfrak{A} \times {}^p\mathfrak{A} \end{aligned}$$

used to define the product $f_1 * f_2$ in $H({}^p\mathcal{A})$. It suffices to show that the fibre product

$$T = (S_1 \times S_2) \times_{{}^p\mathfrak{A} \times {}^p\mathfrak{A}} {}^p\mathfrak{A}^{(2)}$$

is a scheme. Consider the open cover $\{{}^p\mathfrak{A}_n\}_n$ of ${}^p\mathfrak{A}$ given in Proposition 1.5. The first thing we notice is that the collection $\{{}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n\}_n$ is an open cover of ${}^p\mathfrak{A} \times {}^p\mathfrak{A}$ (it covers the whole product via Lemma 1.6). Pulling it back via $f_1 \times f_2$ yields open covers $\{S_{i,n}\}_n$ for each of the S_i and an open cover $\{S_{1,n} \times S_{2,n}\}_n$ of $S_1 \times S_2$.

On the other hand, by the proof of Proposition 2.6 we have an open cover $\{{}^p\mathfrak{A}_n^{(2)}\}_n$ of ${}^p\mathfrak{A}^{(2)}$. By pulling back we obtain an open cover $\{T_n\}_n$ of T . By chasing around base-changes one can see that

$$T_n = (S_{1,n} \times S_{2,n}) \times_{{}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n} {}^p\mathfrak{A}_n^{(2)}.$$

The functor $\widetilde{\tau}_n$ induces morphisms ${}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n \rightarrow \mathfrak{A} \times \mathfrak{A}$, ${}^p\mathfrak{A}_n^{(2)} \rightarrow \mathfrak{A}^{(2)}$ and it is easy to check that

$${}^p\mathfrak{A}_n^{(2)} = ({}^p\mathfrak{A}_n \times {}^p\mathfrak{A}_n) \times_{\mathfrak{A} \times \mathfrak{A}} \mathfrak{A}^{(2)}$$

thus $T_n = (S_{1,n} \times S_{2,n}) \times_{\mathfrak{A} \times \mathfrak{A}} \mathfrak{A}^{(2)}$ and by [Bri10b, Theorem 5.1] it is a scheme. We conclude that T is also scheme. For $p = 0$ one can pass over to X^+ and repeat the same argument. ■

We now briefly turn back to the case of coherent sheaves. The *semi-classical* Hall algebra of coherent sheaves $H_{\text{sc}}(\mathcal{A})$ (defined as $H_{\text{reg}}(\mathcal{A})/(\mathbb{L} - 1)H_{\text{reg}}(\mathcal{A})$, analogously as above) can be equipped with a Poisson bracket given by

$$\{f, g\} = \frac{f * g - g * f}{\mathbb{L} - 1}.$$

There is another Poisson algebra $\mathcal{Q}_\sigma[\Gamma]$, which depends on a choice $\sigma \in \{-1, 1\}$, defined as the \mathbb{Q} -vector space spanned by symbols q^α , with $\alpha \in {}^p\Gamma$, together with a product

$$q^{\alpha_1} * q^{\alpha_2} = \sigma^{\chi(\alpha_1, \alpha_2)} q^{\alpha_1 + \alpha_2}.$$

and a Poisson bracket

$$\{q^{\alpha_1}, q^{\alpha_2}\} = \sigma^{\chi(\alpha_1, \alpha_2)} \chi(\alpha_1, \alpha_2) q^{\alpha_1 + \alpha_2} = \chi(\alpha_1, \alpha_2) (q^{\alpha_1} * q^{\alpha_2}).$$

Given a locally constructible function [JS08, Chapter 2] $\lambda : \mathfrak{A}(\mathbb{C}) \rightarrow \mathbb{Z}$, satisfying the assumptions in [Bri10b, Theorem 5.2], there exists a morphism of Poisson algebras

$$I : H_{\text{sc}}(\mathcal{A}) \rightarrow \mathcal{Q}_\sigma[\Gamma]$$

called the *integration morphism* and such that if $f : W \rightarrow \mathfrak{A}$ is a map from a variety factoring through \mathfrak{A}_α then

$$I([W \rightarrow \mathfrak{A}_\alpha \subset \mathfrak{A}]) = \chi_{\text{top}}(W, f^* \lambda) q^\alpha$$

where

$$\chi_{\text{top}}(W, f^* \lambda) = \sum_{n \in \mathbb{Z}} n \chi_{\text{top}}((\lambda \circ f)^{-1}(n))$$

and where, for a variety Z , $\chi_{\text{top}}(Z)$ denotes the topological Euler characteristic of the associated complex space $Z(\mathbb{C})$.

For $\sigma = 1$ one can choose λ to be identically equal to 1. This gives a well-defined integration morphism which in turn leads to *naive* curve counting invariants. We are more interested in the case $\sigma = -1$ (although what follows certainly holds for the naive invariants as well) where one takes Behrend's microlocal function ν . For $H_{\text{sc}}(\mathcal{A})$ we know [JS08, Theorem 5.5] that the Behrend function satisfies the necessary hypotheses and thus yields an integration morphism.

To define an integration morphism in the context of perverse coherent sheaves we first define $\mathbf{Q}_{\sigma}[\Gamma]$ analogously as $\mathbf{Q}_{\sigma}[\Gamma]$, but using the effective cone of perverse coherent sheaves. In this context, we may still use Behrend's function. More precisely, every Artin stack \mathfrak{M} locally of finite type over \mathbb{C} comes equipped with a Behrend function $\nu_{\mathfrak{M}}$ and given any smooth morphism $f : \mathfrak{M}' \rightarrow \mathfrak{M}$ of relative dimension d we have $f^* \nu_{\mathfrak{M}} = (-1)^d \nu_{\mathfrak{M}'}$. To obtain an integration morphism on $H(\mathcal{A})$ the Behrend function must satisfy the assumptions of [Bri10b, Theorem 5.2]. But these concern only the *points* of ${}^p\mathfrak{A}$ and we know that ${}^p\mathfrak{A}$ is locally isomorphic to \mathfrak{A} (or \mathfrak{A}^+ , for $p = 0$), so the assumptions are satisfied and we have a well-defined integration morphism

$$I : H(\mathcal{A}) \rightarrow \mathbf{Q}_{\sigma}[\Gamma].$$

As a last comment, we point out that ${}^p\mathfrak{T}$ (defined in the previous section) is an open substack of both ${}^p\mathfrak{A}$ and of \mathfrak{A} . As open immersion are smooth of relative dimension zero, the pullbacks of $\nu_{p\mathfrak{A}}$ and $\nu_{\mathfrak{A}}$ to ${}^p\mathfrak{T}$ coincide with $\nu_{p\mathfrak{T}}$. This ensures that no discrepancies occur when defining DT invariants in the different ways in the sequel.

3. IDENTITIES

Again, we assume to be working in Situation 1.1. We redraw the relevant diagram.

$$\begin{array}{ccc} X & \xleftarrow{\phi} & X^+ \\ & \searrow f & \swarrow f^+ \\ & Y & \end{array}$$

Here X and X^+ are smooth and projective threefolds with trivial canonical bundle satisfying $H^1(X, \mathcal{O}_X) = 0 = H^1(X^+, \mathcal{O}_{X^+})$ and f^+ is the flop of f . Recall that we denote by $\mathcal{A}, \mathcal{A}^+$ the categories of coherent sheaves of X and X^+ respectively.

In the previous sections we reminded ourselves of the derived equivalence $\Phi : D^b(X^+) \rightarrow D^b(X)$, of the categories of perverse coherent sheaves ${}^p\mathcal{A}, {}^p\mathcal{A}^+$ (where $p = -1, 0$) and of the subcategories ${}^p\mathcal{T}, {}^p\mathcal{F}$. The key fact is that Φ swaps the categories of perverse coherent sheaves, namely $\Phi({}^q\mathcal{A}^+) = {}^p\mathcal{A}$, with $q = -(p+1)$. We also reminded ourselves of the motivic Hall algebra of coherent sheaves $H(\mathcal{A})$, defined as the Grothendieck ring $K(\text{St}/\mathfrak{A})$ of stacks over the stack of coherent sheaves \mathfrak{A} equipped with a convolution product. We also constructed a moduli stack ${}^p\mathfrak{A}$ parameterising objects in ${}^p\mathcal{A}$ and the Hall algebra $H(\mathcal{A})$ of perverse coherent sheaves, together with the subalgebra of regular elements $H_{\text{reg}}({}^p\mathcal{A})$, its semi-classical limit $H_{\text{sc}}({}^p\mathcal{A})$ and the integration morphism $I : H_{\text{sc}}({}^p\mathcal{A}) \rightarrow \mathbf{Q}_{\sigma}[\Gamma]$. Recall that ${}^p\Gamma$ is the effective cone of perverse coherent sheaves sitting inside the numerical Grothendieck group $N(X)$ and we take $\sigma = -1, 1$ depending on the choice of a locally constructible function on ${}^p\mathfrak{A}$ (either the function identically equal to one or the Behrend function).

3.1. A Route. Before going into the technical details we would like to give a moral proof our main result (Theorem 3.27) which will later guide us through the maze of technical details. The two key results are the identities (3.1), (3.2).

As remarked in the introduction, we think of Donaldson-Thomas (DT) invariants as weighted Euler characteristics of the Hilbert scheme. As the structure sheaf is a perverse coherent sheaf it makes sense to speak about a *perverse Hilbert scheme* ${}^p\text{Hilb}_X$ parameterising quotients in ${}^p\mathcal{A}$. We define *relative* DT invariants $\text{DT}_{X/Y}$ to be the weighted Euler characteristic of ${}^p\text{Hilb}_X$. Whatever $\text{DT}_{X/Y}$ may be it has the

advantage of being invariant under the flop (via the derived equivalence Φ) in a sense to be made precise below. In symbols this becomes $\mathrm{DT}_{X/Y} = \phi_* \mathrm{DT}_{X^+/Y}$.

Our goal is to rewrite $\mathrm{DT}_{X/Y}$ in more familiar terms, which is where the Hall algebra comes into play. The Hilbert scheme Hilb_X maps to \mathfrak{A} by taking a quotient $\mathcal{O}_X \rightarrow E$ to E , thus we have an element $\mathcal{H} \in H(\mathcal{A})$.⁵ From the previous section we know that the integration morphism is related to taking weighted Euler characteristics and in fact integrating \mathcal{H} gives the generating series for the DT invariants⁶

$$I(\mathcal{H}) \text{ “=” } \mathrm{DT}_X := \sum_{\beta, n} (-1)^n \mathrm{DT}_X(\beta, n) q^{(\beta, n)}$$

where $\beta \in N_1(X)$ ranges among curve-classes in X and $n \in \mathbb{Z}$ is a zero-cycle. The perverse Hilbert scheme ${}^p\mathrm{Hilb}_X$ produces a corresponding element ${}^p\mathcal{H}$ of $H({}^p\mathcal{A})$, which upon being integrated produces $\mathrm{DT}_{X/Y}$.

The first thing we remark is that, as a quotient (in \mathcal{A}) of \mathcal{O}_X lies in ${}^p\mathcal{T}$ and ${}^p\mathcal{T} \subset {}^p\mathcal{A}$, we can interpret \mathcal{H} as an element of $H({}^p\mathcal{A})$. There is an element $1_{{}^p\mathcal{F}[1]}$ in $H({}^p\mathcal{A})$ represented by the inclusion ${}^p\mathfrak{F}[1] \subset {}^p\mathfrak{A}$. There is also a stack parameterising objects of ${}^p\mathcal{F}[1]$ together with a morphism from \mathcal{O}_X . This stack maps down to ${}^p\mathfrak{A}$ by forgetting the morphism, yielding an element $1_{{}^p\mathcal{F}[1]}^{\mathcal{O}}$. We will prove that there is an identity

$$(3.1) \quad {}^p\mathcal{H} * 1_{{}^p\mathcal{F}[1]} = 1_{{}^p\mathcal{F}[1]}^{\mathcal{O}} * \mathcal{H}$$

in the Hall algebra of perverse coherent sheaves. Let us see how one might deduce this.

We extend the notation $1_{{}^p\mathcal{F}[1]}, 1_{{}^p\mathcal{F}[1]}^{\mathcal{O}}$ to general subcategories $\mathcal{B} \subset {}^p\mathcal{A}$ (whenever it makes sense) producing elements $1_{\mathcal{B}}, 1_{\mathcal{B}}^{\mathcal{O}}$ in $H({}^p\mathcal{A})$, and similarly for $H(\mathcal{A})$. As $({}^p\mathcal{T}, {}^p\mathcal{F})$ is a torsion pair in \mathcal{A} , we have an identity $1_{\mathcal{A}} = 1_{{}^p\mathcal{T}} * 1_{{}^p\mathcal{F}}$. This follows from the fact that for any coherent sheaf E there is a unique exact sequence $T \hookrightarrow E \rightarrow F$ with $T \in {}^p\mathcal{T}$, $F \in {}^p\mathcal{F}$. Notice that the product $1_{{}^p\mathcal{T}} * 1_{{}^p\mathcal{F}}$ is given by $[Z \rightarrow \mathfrak{A}]$ where Z parameterises exact sequences $T \hookrightarrow E \rightarrow F$ and the morphism sends such an exact sequence to E .

We also have an identity $1_{\mathcal{A}}^{\mathcal{O}} = 1_{{}^p\mathcal{T}}^{\mathcal{O}} * 1_{{}^p\mathcal{F}}^{\mathcal{O}}$. This is a consequence of the previous identity plus the fact that $\mathrm{Hom}(\mathcal{O}_X, F) = 0$ (Lemma 1.3). This last fact also tells us that $1_{{}^p\mathcal{F}}^{\mathcal{O}} = 1_{{}^p\mathcal{F}}$. Moreover, the first isomorphism theorem for the abelian category \mathcal{A} is reflected in the identity $1_{\mathcal{A}}^{\mathcal{O}} = \mathcal{H} * 1_{\mathcal{A}}$ (any morphism $\mathcal{O}_X \rightarrow E$ factors through its image). Combining everything together we see that $\mathcal{H} = 1_{\mathcal{A}}^{\mathcal{O}} * 1_{\mathcal{A}}^{-1} = 1_{{}^p\mathcal{T}}^{\mathcal{O}} * 1_{{}^p\mathcal{T}}^{-1}$.

A parallel argument can be carried over for ${}^p\mathcal{A}$ yielding ${}^p\mathcal{H} = 1_{{}^p\mathcal{A}}^{\mathcal{O}} * 1_{{}^p\mathcal{A}}^{-1} = 1_{{}^p\mathcal{F}[1]}^{\mathcal{O}} * (1_{{}^p\mathcal{T}}^{\mathcal{O}} * 1_{{}^p\mathcal{T}}^{-1}) * 1_{{}^p\mathcal{F}[1]}^{-1} = 1_{{}^p\mathcal{F}[1]}^{\mathcal{O}} * {}^p\mathcal{H} * 1_{{}^p\mathcal{F}[1]}^{-1}$ from which we extract (3.1) (for the identity $1_{{}^p\mathcal{A}}^{\mathcal{O}} = 1_{{}^p\mathcal{F}[1]}^{\mathcal{O}} * 1_{{}^p\mathcal{T}}^{\mathcal{O}}$ one uses the fact $\mathrm{Hom}_X(\mathcal{O}_X, F[2]) = 0$, found in Lemma 1.3).

We now want to understand how to rewrite $1_{{}^p\mathcal{F}[1]}^{\mathcal{O}}$ in a more familiar form. It turns out that duality interchanges ${}^q\mathcal{T}$ and ${}^p\mathcal{F}$. Precisely, let \mathcal{Q} be the subcategory \mathcal{A} consisting of sheaves with no subsheaves supported in dimension zero. Let \mathcal{Q}_{\bullet} denote the subcategory of \mathcal{Q} made up of sheaves supported on the exceptional locus of the flopping contraction f and let ${}^p\mathcal{T}_{\bullet} = \mathcal{Q}_{\bullet} \cap {}^p\mathcal{T}$. It is a simple computation (Lemma 3.11) to check that the duality functor $\mathbb{D} = \mathrm{R}\underline{\mathrm{Hom}}_X(-, \mathcal{O}_X)[2]$ takes ${}^q\mathcal{T}_{\bullet}$ to ${}^p\mathcal{F}$. The category \mathcal{Q} is related to DT invariants in the following way.

There is an identity $1_{\mathcal{Q}}^{\mathcal{O}} = \mathcal{H}^{\#} * 1_{\mathcal{Q}}$ in $H(\mathcal{A})$, where $\mathcal{H}^{\#}$ corresponds to (yet another) Hilbert scheme of a tilt $\mathcal{A}^{\#}$ of \mathcal{A} . We can restrict to sheaves supported on the exceptional locus of f , which yields an identity $1_{\mathcal{Q}_{\bullet}}^{\mathcal{O}} = \mathcal{H}_{\bullet}^{\#} * 1_{\mathcal{Q}_{\bullet}}$, which can be refined to $1_{{}^p\mathcal{T}_{\bullet}}^{\mathcal{O}} = \mathcal{H}_{\bullet}^{\#} * 1_{{}^p\mathcal{T}_{\bullet}}$. Integrating $\mathcal{H}_{\bullet}^{\#}$ gives the generating series for the Pandharipande-Thomas (PT) invariants of X [Bri10a, Lemma 5.5]

$$I(\mathcal{H}_{\bullet}^{\#}) = \mathrm{PT}_f := \sum_{\substack{\beta, n \\ f_*\beta=0}} (-1)^n \mathrm{PT}_X(\beta, n) q^{(\beta, n)}$$

⁵Strictly speaking this is false as Hilb_X is not of finite type. We shall later enlarge our Hall algebra precisely to deal with this issue.

⁶Again, this is a small lie. We should really restrict to the Hilbert scheme parameterising quotients supported in dimensions less than one. However, for the remainder of this subsection we shall not concern ourselves with these matters.

where β ranges over the curve-classes contracted by f . We mention in passing that the PT invariants of X are (conjecturally [MNOP]) related to the Gromov-Witten invariants of X . If we let

$$\mathrm{DT}_{X,0} := \sum_n (-1)^n \mathrm{DT}_X(0, n) q^n$$

we know [Bri10b, Theorem 1.1] that the *reduced* DT invariants $\mathrm{DT}'_X := \mathrm{DT}_X / \mathrm{DT}_{X,0}$ coincide with the PT invariants PT_X .

Now, the (shifted) duality functor $\mathbb{D}' = \mathbb{D}[1]$ induces a morphism between Hall algebras⁷ and takes ${}^q\mathcal{T}_\bullet$ to ${}^p\mathcal{F}[1]$ and so we have $\mathbb{D}'(1_{q\mathcal{T}_\bullet}) = 1_{p\mathcal{F}[1]}$. Furthermore, as a consequence of Serre duality, $\mathbb{D}'(1_{q\mathcal{T}_\bullet}^\mathcal{O}) = 1_{p\mathcal{F}[1]}^\mathcal{O}$. As a result we have

$$(3.2) \quad 1_{p\mathcal{F}[1]}^\mathcal{O} = 1_{p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#)$$

as $1_{p\mathcal{F}[1]}^\mathcal{O} = \mathbb{D}'(1_{q\mathcal{T}_\bullet}^\mathcal{O}) = \mathbb{D}'(\mathcal{H}_\bullet^\# * 1_{q\mathcal{T}_\bullet}) = \mathbb{D}'(1_{q\mathcal{T}_\bullet}) * \mathbb{D}'(\mathcal{H}_\bullet^\#) = 1_{p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#)$ (notice that duality is an *anti*-equivalence and thus swaps extensions). We can rewrite (3.1) as follows.

$$(3.3) \quad {}^p\mathcal{H} * 1_{p\mathcal{F}[1]} = 1_{p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#) * \mathcal{H}$$

Duality and integration can be interchanged up to a flip in signs. Precisely

$$\mathrm{I}(\mathbb{D}'(\mathcal{H}_\bullet^\#)) = \mathrm{PT}_f^\vee := \sum_{\substack{\beta, n \\ f_*\beta=0}} (-1)^n \mathrm{PT}_X(-\beta, n) q^{(\beta, n)}.$$

Upon integrating the two sides of (3.3) the two $1_{p\mathcal{F}[1]}$ cancel out⁸ and we are left with the identity

$$\mathrm{DT}_{X/Y} = \mathrm{PT}_f^\vee \cdot \mathrm{DT}_X.$$

Finally we pass from one side of the flop to the other using the derived equivalence Φ , which is compatible with the strict transform of divisors. Specifically, we have a morphism $\phi_* : N_1(X^+) \rightarrow N_1(X)$ given by the inverse of the transpose of the strict transform of divisors. This morphism extends to the generating series we have been considering by taking $q^{(\beta, n)}$ to $q^{(\phi_*\beta, n)}$.

As we've remarked above, one has $\mathrm{DT}_{X/Y} = \phi_* \mathrm{DT}_{X^+/Y}$, which leads to

$$\mathrm{PT}_f^\vee \cdot \mathrm{DT}_X = \mathrm{DT}_{X/Y} = \phi_* \mathrm{DT}_{X^+/Y} = \phi_* (\mathrm{PT}_{f^+}^\vee \cdot \mathrm{DT}_{X^+}).$$

To prove the identity (\star) of the introduction one notices that $\mathrm{DT}_{X,0}$ is simply given by the topological Euler characteristic $\chi_{\mathrm{top}}(X)$ of X multiplied by the McMahon function $M(q)$ [BF08] and that $\chi_{\mathrm{top}}(X) = \chi_{\mathrm{top}}(X^+)$ [Bat99]. Using the identity above we indeed deduce (\star) .

$$\mathrm{DT}_f^\vee \cdot \mathrm{DT}_X = \phi_* (\mathrm{DT}_{f^+}^\vee \cdot \mathrm{DT}_{X^+})$$

3.2. The Perverse Hilbert Scheme. We now proceed along the route traced in the previous subsection, but taking care of technical details. Let us start by working in *infinite-type* versions $H_\infty(\mathcal{A})$, $H_\infty({}^p\mathcal{A})$ of our Hall algebras. The advantage of H_∞ is that we include stacks *locally* of finite type over \mathbb{C} (e.g. ${}^p\mathfrak{A}$), the disadvantage is that we do not have an integration morphism at our disposal. To define this algebra we proceed exactly as in the previous section: the only differences being that we allow our stacks to be *locally* of finite type over \mathbb{C} , we insist that geometric bijections be finite type morphisms and we disregard the disjoint union relation.⁹

The first element we consider is $\mathcal{H} \in H_\infty(\mathcal{A})$ corresponding to the Hilbert scheme of X , which parameterises quotients of \mathcal{O}_X in \mathcal{A} . To be precise, \mathcal{H} is represented by the forgetful morphism $\mathrm{Hilb}_X \rightarrow \mathfrak{A}$, which takes a quotient $\mathcal{O}_X \twoheadrightarrow E$ to E . For us, the important thing to notice is that if $\mathcal{O}_X \twoheadrightarrow E$ is a quotient in \mathcal{A} , then $E \in {}^p\mathcal{T}$. This is a consequence of $\mathcal{O}_X \in {}^p\mathcal{T}$ and of the fact that the torsion part of a torsion pair is closed under quotients. Thus the morphism $\mathrm{Hilb}_X \rightarrow \mathfrak{A}$ factors through ${}^p\mathfrak{T}$. As ${}^p\mathfrak{T} \subset {}^p\mathfrak{A}$, \mathcal{H} can be interpreted as an element of $H_\infty({}^p\mathcal{A})$.

⁷More precisely it induces a morphism between certain subalgebras to be defined below.

⁸This is the content of Proposition 3.23, a consequence of a powerful result by Joyce.

⁹If we allowed both the disjoint union relation and spaces of infinite type then we would be left with the zero ring. Indeed, if Z is an infinite disjoint union of points, then $Z \setminus \{pt\} \cong Z$ and thus $[Z] = [Z] - 1$ so $1 = 0$. The finite type assumption for geometric bijections is there to avoid pathologies such as an infinite disjoint union of points representing the same class as a line.

Once and for all we establish some general notation. For $\mathcal{B} \subset \mathcal{A}$ a subcategory we denote $1_{\mathcal{B}}$ the element of $H_{\infty}(\mathcal{A})$ represented by the inclusion of stacks $\mathfrak{B} \subset \mathfrak{A}$ (when this is an open immersion). Another important stack is $\mathfrak{A}^{\mathcal{O}}$, the stack of *framed* coherent sheaves [Bri10a, Section 2.3], which parameterises sheaves with a fixed global section $\mathcal{O}_X \rightarrow E$. By considering surjective sections we can realise Hilb_X as an open subscheme of $\mathfrak{A}^{\mathcal{O}}$. We have a forgetful map $\mathfrak{A}^{\mathcal{O}} \rightarrow \mathfrak{A}$, which takes a morphism $\mathcal{O}_X \rightarrow E$ to E . Given an open substack $\mathfrak{B} \subset \mathfrak{A}$, we can consider the fibre product $\mathfrak{B}^{\mathcal{O}} = \mathfrak{B} \times_{\mathfrak{A}} \mathfrak{A}^{\mathcal{O}}$, which gives an element $1_{\mathfrak{B}}^{\mathcal{O}} \in H_{\infty}(\mathcal{A})$.

We want to emulate this last construction for $H_{\infty}({}^p\mathcal{A})$. We first need to construct a stack of *framed perverse coherent sheaves* ${}^p\mathfrak{A}^{\mathcal{O}}$. Before we do that, notice that we also have a stack \mathfrak{C} parameterising coherent sheaves on Y and a corresponding stack of framed sheaves $\mathfrak{C}^{\mathcal{O}}$. For $P \in {}^p\mathcal{A}$, morphisms $\mathcal{O}_X \rightarrow P$ correspond (by adjunction) to morphisms $\mathcal{O}_Y \rightarrow Rf_*P$. We know that Rf_*P is a sheaf, so morphisms $\mathcal{O}_X \rightarrow P$ correspond to points of $\mathfrak{C}^{\mathcal{O}}$.

To make this argument work in families we observe that the pushforward Rf_* induces a morphism of stacks ${}^p\mathfrak{A} \rightarrow \mathfrak{C}$. Indeed, if P is a family of perverse coherent sheaves over a base S then $Rf_{S,*}P$ is a family of coherent sheaves on Y over S . This can be seen by taking a point $s \in S$ and pulling back to fibres: by base-change $(Rf_{S,*}P)|_{Y_s}^L = Rf_{s,*}(P|_{X_s}^L)$, which is a sheaf. We can thus realise ${}^p\mathfrak{A}^{\mathcal{O}}$ as the pullback of the following diagram.

$$\begin{array}{ccc} & \mathfrak{C}^{\mathcal{O}} & \\ & \downarrow & \\ {}^p\mathfrak{A} & \longrightarrow & \mathfrak{C} \end{array}$$

We have elements $1_{{}^p\mathcal{F}[1]}, 1_{{}^p\mathcal{T}} \in H_{\infty}({}^p\mathcal{A})$ corresponding to the subcategories ${}^p\mathcal{F}[1], {}^p\mathcal{T}$ of ${}^p\mathcal{A}$. By taking fibre products with ${}^p\mathfrak{A}^{\mathcal{O}} \rightarrow {}^p\mathcal{A}$ we produce elements $1_{{}^p\mathcal{F}[1]}^{\mathcal{O}}, 1_{{}^p\mathcal{T}}^{\mathcal{O}} \in H_{\infty}({}^p\mathcal{A})$.

We also want a *perverse Hilbert scheme* ${}^p\text{Hilb}_{X/Y}$ of X over Y parameterising quotients of \mathcal{O}_X in ${}^p\mathcal{A}$. One can realise it as an open substack of ${}^p\mathfrak{A}^{\mathcal{O}}$. Indeed, for $\alpha : \mathcal{O}_X \rightarrow P$ with $P \in {}^p\mathcal{A}$, being surjective is equivalent to the cone of α lying in ${}^p\mathcal{A}[1]$, which we know from the first section to be an open condition on ${}^p\mathfrak{A}^{\mathcal{O}}$. Thus we have an element ${}^p\mathcal{H} \in H_{\infty}({}^p\mathcal{A})$.

3.3. A First Identity. We want to prove the identity

$$(3.4) \quad {}^p\mathcal{H} * 1_{{}^p\mathcal{F}[1]} = 1_{{}^p\mathcal{F}[1]}^{\mathcal{O}} * {}^p\mathcal{H}$$

which we motivated in the beginning of this section. The left hand side is represented by a stack \mathfrak{M}_L , parameterising diagrams

$$\begin{array}{ccc} \mathcal{O}_X & & \\ \downarrow & & \\ P_1 & \hookrightarrow E \twoheadrightarrow & P_2 \end{array}$$

where all objects are in ${}^p\mathcal{A}$, the sequence $P_1 \hookrightarrow E \twoheadrightarrow P_2$ is exact in ${}^p\mathcal{A}$, $\mathcal{O}_X \twoheadrightarrow P_1$ is surjective in ${}^p\mathcal{A}$ and $P_2 \in {}^p\mathcal{F}[1]$.¹⁰

The right hand side is represented by a stack \mathfrak{M}_R parameterising diagrams

$$\begin{array}{ccc} \mathcal{O}_X & & \mathcal{O}_X \\ \downarrow & & \downarrow \text{sur} \\ F[1] & \hookrightarrow E \twoheadrightarrow & T \end{array}$$

where the horizontal maps form a short exact sequence in ${}^p\mathcal{A}$, $F \in {}^p\mathcal{F}, T \in {}^p\mathcal{T}$ and the map $\mathcal{O}_X \rightarrow T$ is surjective as a morphism in \mathcal{A} . We remind ourselves that $({}^p\mathcal{F}[1], {}^p\mathcal{T})$ is a torsion pair in ${}^p\mathcal{A}$ so that given a perverse sheaf E , there is a unique exact sequence $F[1] \hookrightarrow E \twoheadrightarrow T$, with $F \in {}^p\mathcal{F}, T \in {}^p\mathcal{T}$.

¹⁰To be precise, over a base U , the groupoid $\mathfrak{M}_L(U)$ consists of diagrams as above which, upon restricting to fibres of points of U , satisfy the required properties. Similar remarks will be implicit for the other stacks we define below.

We shall make use of the following lemma (by *perverse* kernel, cokernel, surjection etc. we mean kernel, cokernel, surjection etc. in the abelian category ${}^p\mathcal{A}$).

3.5 Lemma. Let $\varphi : \mathcal{O}_X \rightarrow E$ be a morphism from the structure sheaf to a perverse coherent sheaf. Then the perverse cokernel of φ lies in ${}^p\mathcal{F}[1]$ if and only if the cone of φ belongs to $D^{\leq -1}(X)$.

Proof: Let $\sigma : E \rightarrow T$ be the surjection from E to its torsion-free part. We first show that the statement ${}^p\text{coker } \varphi \in {}^p\mathcal{F}[1]$ is equivalent to $\sigma\varphi$ being surjective as a morphism of coherent sheaves. First of all notice that as $H^0(\sigma)$ is an isomorphism then $H^0(\sigma\varphi)$ is surjective if and only if $H^0(\varphi)$ is surjective. Consider the diagram obtained by factoring φ through the perverse image and by taking the perverse cokernel.

$$\begin{array}{ccccc} \mathcal{O}_X & \xrightarrow{\varphi} & E & & \\ & \searrow \alpha & \nearrow \beta & \searrow & \\ & I & & K & \end{array}$$

Glancing at the cohomology sheaves long exact sequence we see that $H^0(\alpha)$ is surjective. Thus $H^0(\varphi)$ is surjective if and only if $H^0(\beta)$ is surjective if and only if $H^0(K) = 0$ if and only if $K \in {}^p\mathcal{F}[1]$.

Let now C be the cone of φ . By taking the cohomology sheaves long exact sequence we immediately see that $H^0(\varphi)$ is surjective if and only if $C \in D^{\leq -1}(X)$. \blacksquare

We now define a stack \mathfrak{M}' parameterising diagrams of the form

$$\begin{array}{c} \mathcal{O}_X \\ \downarrow \varphi \\ E \end{array}$$

where ${}^p\text{coker } \varphi \in {}^p\mathcal{F}[1]$. By the previous lemma this last condition is equivalent to $\text{cone}(\varphi) \in D^{\leq -1}(X)$, which is open. Thus \mathfrak{M}' is an open substack of the stack of framed perverse sheaves ${}^p\mathfrak{A}^{\mathcal{O}}$.

3.6 Proposition. There is a map $\mathfrak{M}_L \rightarrow \mathfrak{M}'$ induced by the composition $\mathcal{O}_X \twoheadrightarrow P_1 \hookrightarrow E$. This map is a geometric bijection.

Proof: In view of the previous lemma this map induces an equivalence on \mathbb{C} -points. To prove that it is of finite type we use a fact that shall be proved later. Namely, for a fixed numerical class $\alpha \in N(X)$, the open substack $\mathfrak{M}_{L,\alpha}$ parameterising diagrams with $[E] = \alpha$ is of finite type. By restriction we have a map between $\mathfrak{M}_{L,\alpha}$ and \mathfrak{M}'_{α} which is of finite type as $\mathfrak{M}_{L,\alpha}$ is of finite type. Thus the morphism $\mathfrak{M}_L \rightarrow \mathfrak{M}'$ is of finite type. \blacksquare

We define another stack \mathfrak{M} parameterising diagrams of the form

$$\begin{array}{ccccc} & \mathcal{O}_X & & & \\ & \downarrow \varphi & & & \\ F[1] & \hookrightarrow & E & \twoheadrightarrow & T \end{array}$$

where the horizontal maps form a short exact sequence of perverse sheaves, $F \in {}^p\mathcal{F}$, $T \in {}^p\mathcal{T}$ and ${}^p\text{coker } \varphi \in {}^p\mathcal{F}[1]$. This stack can be obtained as a fibre product as follows. The element $1_{{}^p\mathcal{F}[1]} * 1_{{}^p\mathcal{T}}$ is represented by a morphism $Z \rightarrow {}^p\mathfrak{A}$ and \mathfrak{M} is the top left corner of the following cartesian diagram.

$$\begin{array}{ccc} \mathfrak{M} & \longrightarrow & \mathfrak{M}' \\ \downarrow & & \downarrow \\ Z & \longrightarrow & {}^p\mathfrak{A} \end{array}$$

3.7 Proposition. The morphism $\mathfrak{M} \rightarrow \mathfrak{M}'$ defined by forgetting the exact sequence is a geometric bijection.

Proof: The morphism in question is precisely the top row of the previous diagram. The bottom row is obtained by composing the top arrows of the following diagram.

$$\begin{array}{ccccc} Z & \longrightarrow & {}^p\mathfrak{A}^{(2)} & \xrightarrow{b} & {}^p\mathfrak{A} \\ \downarrow & & \downarrow & & \\ {}^p\mathfrak{F}[1] \times {}^p\mathfrak{T} & \longrightarrow & {}^p\mathfrak{A} \times {}^p\mathfrak{A} & & \end{array}$$

where the bottom row is an open immersion (and thus of finite type) and the morphism b is of finite type (this follows from the fact that b locally is isomorphic to the analogous morphism for coherent sheaves). The morphism $Z \rightarrow {}^p\mathfrak{A}$ induces an equivalence on \mathbb{C} -points because $({}^p\mathcal{F}[1], {}^p\mathcal{T})$ is a torsion pair in ${}^p\mathcal{A}$ (and thus any perverse coherent sheaf has a unique short exact sequence with torsion kernel and torsion-free cokernel) and because an automorphism of a short exact sequence which is the identity on the middle term is trivial. As $\mathfrak{M} \rightarrow \mathfrak{M}'$ is a base change of $Z \rightarrow {}^p\mathfrak{A}$ we are done. \blacksquare

Thus the identity (3.4) boils down to proving that \mathfrak{M} and \mathfrak{M}_R represent the same element in $H_\infty({}^p\mathcal{A})$. To do this we use one last stack \mathfrak{N} and build a pair of Zariski fibrations with same fibres. We define the stack \mathfrak{N} to be the moduli of the following diagrams

$$\begin{array}{ccc} & & \mathcal{O}_X \\ & & \downarrow \text{sur} \\ F[1] \hookrightarrow E \twoheadrightarrow T \end{array}$$

where the horizontal maps form a short exact sequence of perverse sheaves, $F \in {}^p\mathcal{F}$, $T \in {}^p\mathcal{T}$ and the map $\mathcal{O}_X \rightarrow T$ is surjective on H^0 . This stack is also a fibre product of known stacks (compare with the element $1_{{}^p\mathcal{F}[1]} * \mathcal{H}$). Notice that there are two maps $\mathfrak{M} \rightarrow \mathfrak{N} \leftarrow \mathfrak{M}_R$. The map $\mathfrak{M}_R \rightarrow \mathfrak{N}$ is given by forgetting the morphism $\mathcal{O}_X \rightarrow F[1]$. The map $\mathfrak{M} \rightarrow \mathfrak{N}$ is given by composition $\mathcal{O}_X \rightarrow E \rightarrow T$.

3.8 Proposition. The maps $\mathfrak{M} \rightarrow \mathfrak{N} \leftarrow \mathfrak{M}_R$ are two Zariski fibrations with same fibres.

Proof: The idea is that over a perverse coherent sheaf E the morphism $\mathfrak{M}_R \rightarrow \mathfrak{N}$ has fibres $\text{Hom}_X(\mathcal{O}_X, F[1])$ while $\mathfrak{M} \rightarrow \mathfrak{N}$ has fibres lifts $\mathcal{O}_X \rightarrow E$. The long exact sequence

$$0 \rightarrow \text{Hom}_X(\mathcal{O}_X, F[1]) \rightarrow \text{Hom}_X(\mathcal{O}_X, E) \rightarrow \text{Hom}_X(\mathcal{O}_X, T) \rightarrow 0$$

(where that the last zero follows from the fact that F is supported in dimension one) tells us that given a choice of a lift of $\mathcal{O}_X \rightarrow T$ all lifts are in bijection with $\text{Hom}_X(\mathcal{O}_X, F[1])$.

Let's see how to make this argument work in families. Let $S \rightarrow \mathfrak{N}$ correspond to a diagram of quasi-coherent sheaves on X_S

$$\begin{array}{ccc} & & \mathcal{O}_X \\ & & \downarrow \text{sur} \\ F[1] \hookrightarrow E \twoheadrightarrow T \end{array}$$

with S a connected affine scheme. First of all notice that base change and Lemma 1.3 (and the proof of Proposition A.7) tell us that $Rp_{S,*}F$ is just $H^1(X_S, F)$ shifted by one, where $p_S : X_S \rightarrow S$ is the projection. In addition, $H^1(X_S, F)$ is flat over S , or in other words \mathcal{O}_{X_S} and F have constant Ext groups in the sense of [Bri10b, Section 6.1] (all the others vanish).

Let W be the fibre product $\mathfrak{M}_R \times_{\mathfrak{N}} S$. This is actually a functor which associates to an affine S -scheme $q : T \rightarrow S$ the group $H^1(X_T, q_X^* F)$ and we know by loc. cit. that it is represented by a vector bundle over S of rank the rank of $H^1(X_S, F)$.

Similarly, the fibre product $\mathfrak{M} \times_{\mathfrak{N}} S$ is represented by an affine bundle of rank the rank of $H^1(X_S, F)$ (notice that because of the previous arguments the exact sequence at the beginning of the proof still holds over S). This concludes the proof. \blacksquare

3.4. PT Invariants. We are still left with the task of understanding what we obtain by integrating ${}^p\mathcal{H}$. To achieve this goal we first substitute $1_{{}^p\mathcal{F}[1]}$ with something more recognisable (from the point of view of

the integration morphism I). Recall [Bri10a, Section 2.2] that on \mathcal{A} there is a torsion pair $(\mathcal{P}, \mathcal{Q})$, where \mathcal{P} consists of sheaves supported in dimension zero and \mathcal{Q} is the right orthogonal of \mathcal{P} . In particular, an element $Q \in \mathcal{Q}$ which is supported in dimension one is pure. Notice also that $\mathcal{O}_X \in \mathcal{Q}$. We denote by $\mathcal{A}^\#$ the tilt with respect to $(\mathcal{P}, \mathcal{Q})$, but with the convention

$$\mathcal{P}[-1] \subset \mathcal{A}^\# \subset D^{[0,1]}(X).$$

There exists a scheme $\text{Hilb}_X^\#$ parameterising quotients of \mathcal{O}_X in $\mathcal{A}^\#$. Using [Bri10a, Lemma 2.3] one constructs an element $\mathcal{H}^\# \in H_\infty(\mathcal{A})$ which eventually leads to the PT invariants of X . We recall that quotients of \mathcal{O}_X in $\mathcal{A}^\#$ are exactly morphisms $\mathcal{O}_X \rightarrow Q$, with cokernel in \mathcal{P} and $Q \in \mathcal{Q}$.

In $H_\infty(\mathcal{A})$ we have an element 1_Q given by the inclusion of the stack parameterising objects in \mathcal{Q} inside \mathfrak{A} and its framed version $1_Q^\mathcal{O}$. There is also an identity [Bri10a, Section 4.5]

$$1_Q^\mathcal{O} = \mathcal{H}^\# * 1_Q.$$

We want to restrict the element $\mathcal{H}^\#$ further by considering only quotients supported on the exceptional locus Ex of the flopping contraction $f : X \rightarrow Y$. We thus define the following subcategories.

$$\begin{aligned} \mathcal{Q}_\bullet &= \{Q \in \mathcal{Q} \mid \text{supp } Q \subset \text{Ex}\} \\ {}^p\mathcal{T}_\bullet &= {}^p\mathcal{T} \cap \mathcal{Q}_\bullet. \end{aligned}$$

We can also consider the scheme $\text{Hilb}_{X,\bullet}^\#$ parameterising quotients of \mathcal{O}_X in $\mathcal{A}^\#$ with target supported on Ex (it is indeed an open subscheme of $\text{Hilb}_X^\#$ as we are imposing a restriction on the numerical class of the quotients). From it we obtain an element $\mathcal{H}_\bullet^\# \in H_\infty(\mathcal{A})$. We have the following result.

3.9 Proposition. The following identity in $H_\infty(\mathfrak{A})$ is true.

$$(3.10) \quad 1_{{}^p\mathcal{T}_\bullet}^\mathcal{O} = \mathcal{H}_\bullet^\# * 1_{{}^p\mathcal{T}}.$$

Proof: We start with a remark. If we have a morphism $\mathcal{O}_X \rightarrow T$ in $\mathcal{A}^\#$, with $T \in {}^p\mathcal{T}_\bullet$, we can factor it through its image (in $\mathcal{A}^\#$) $\mathcal{O}_X \rightarrow I \rightarrow T$ and we denote by Q the quotient, again in $\mathcal{A}^\#$. We already know [Bri10a, Lemma 2.3] that I is a sheaf and that the morphism $\mathcal{O}_X \rightarrow I$, as a morphism in \mathcal{A} , has cokernel P supported in dimension zero.

Glancing at the cohomology sheaves long exact sequence reveals that Q is also a sheaf, thus the sequence $I \hookrightarrow T \twoheadrightarrow Q$ is actually a short exact sequence of sheaves. The sheaf Q is in ${}^p\mathcal{T}$, as it is a quotient of T , and it lies in \mathcal{Q} as it is an object of $\mathcal{A}^\#$. Also, Q is supported on Ex as T is, thus $Q \in {}^p\mathcal{T}_\bullet$.

On the other hand, given a morphism of sheaves $\mathcal{O}_X \rightarrow I$, which is an epimorphism in $\mathcal{A}^\#$, and given a short exact sequence $I \hookrightarrow T \twoheadrightarrow Q$, with $I \in \mathcal{Q}_\bullet$ and $Q \in {}^p\mathcal{T}_\bullet$, we claim that $T \in {}^p\mathcal{T}_\bullet$. The fact that $T \in \mathcal{Q}_\bullet$ is clear, if we prove that $I \in {}^p\mathcal{T}$ then we are done.

We know there is an exact sequence $\mathcal{O}_X \rightarrow I \twoheadrightarrow P$, with P supported in dimension zero, viz. a skyscraper sheaf. Let $I \twoheadrightarrow F$ be the projection to the torsion-free part of I . The morphism $\mathcal{O}_X \rightarrow I \twoheadrightarrow F$ is zero, as objects of ${}^p\mathcal{F}$ have no sections. Thus there is a morphism $P \rightarrow F$ such that $I \twoheadrightarrow P \rightarrow F$ is equal to $I \twoheadrightarrow F$. As P is a skyscraper sheaf, the morphisms from it are determined on global sections, thus $P \rightarrow F$ is zero, which in turn implies that $I \twoheadrightarrow F$. Thus $F = 0$ and $I \in {}^p\mathcal{T}$.

Using the remark above we can see that there exists a morphism from the stack parameterising diagrams

$$\begin{array}{c} \mathcal{O}_X \\ \downarrow \\ I \hookrightarrow T \twoheadrightarrow Q \end{array}$$

with $\mathcal{O}_X \rightarrow I$ an epimorphism in $\mathcal{A}^\#$, $I \in \mathcal{Q}_\bullet$, $Q \in {}^p\mathcal{T}_\bullet$, to the stack parameterising morphisms $\mathcal{O}_X \rightarrow T$, with $T \in {}^p\mathcal{T}_\bullet$. This morphism induces an equivalence on \mathbb{C} -points and the fact that it is of finite type will follow from Proposition 3.17 and Proposition 3.12. \blacksquare

3.5. Duality. We observe that we already know, cf. [Bri10a, Lemma 5.5], that the element $\mathcal{H}_\bullet^\#$ produces the generating series for the PT invariants of X

$$\text{PT}_f := \sum_{\substack{\beta, n \\ f_*\beta=0}} (-1)^n \text{PT}_X(\beta, n) q^{(\beta, n)}$$

restricted to the curves which get contracted by f .

What comes up in the identity (3.4) is ${}^p\mathcal{F}$. We will see now how to link this category with the PT invariants, via the duality functor.

3.11 Lemma. Let $\mathbb{D} : \mathcal{D}(X) \rightarrow \mathcal{D}(X)$ be the anti-equivalence defined by

$$E \mapsto \mathbb{D}(E) = R\mathbf{Hom}_X(E, \mathcal{O}_X)[2].$$

Then

$$\begin{aligned}\mathbb{D}({}^p\mathcal{T}_\bullet) &= {}^q\mathcal{F} \\ \mathbb{D}({}^p\mathcal{F}) &= {}^q\mathcal{T}_\bullet\end{aligned}$$

where $q = -(p+1)$.

Proof: The shift $[2]$ in the definition of \mathbb{D} is due to the fact we are dealing with pure sheaves supported in codimension two. Indeed, if \mathcal{Q}_1 is the category of pure sheaves supported in dimension one, then $\mathbb{D}(\mathcal{Q}_1) = \mathcal{Q}_1$ [Bri10a, Lemma 5.6].

We now show that $\mathbb{D}(\mathcal{C}) = \mathcal{C}$, where \mathcal{C} is the category of sheaves with vanishing derived pushdown (via f). First of all $\mathcal{C} \subset \mathcal{Q}_1$, as elements of \mathcal{C} are supported in dimension one and are pure (having a zero-dimensional subsheaf implies the existence of global sections, of which elements of \mathcal{C} have none). We are thus only left to check that $Rf_*\mathbb{D}(\mathcal{C}) = 0$.

$$\begin{aligned}Rf_*\mathbb{D}(\mathcal{C}) &= Rf_*R\mathbf{Hom}_X(\mathcal{C}, \mathcal{O}_X)[2] \\ &= Rf_*R\mathbf{Hom}_X(\mathcal{C}, f^!\mathcal{O}_Y)[2] \\ &= R\mathbf{Hom}_Y(Rf_*\mathcal{C}, \mathcal{O}_Y)[2] = 0\end{aligned}$$

Let now $F \in {}^p\mathcal{F}$. We need to check that $R^1f_*\mathbb{D}(F) = 0$.

$$\begin{aligned}R^1f_*\mathbb{D}(F) &= H^1(Rf_*R\mathbf{Hom}_X(F, \mathcal{O}_X)[2]) \\ &= H^3(R\mathbf{Hom}_Y(Rf_*F, \mathcal{O}_Y)) \\ &= H^3(R\mathbf{Hom}_Y(R^1f_*F[-1], \mathcal{O}_Y)) \\ &= \underline{\mathrm{Ext}}_Y^4(R^1f_*F, \mathcal{O}_Y) \\ &= \mathrm{Ext}_Y^4(R^1f_*F, \mathcal{O}_Y) = 0\end{aligned}$$

where the last equality follows from Serre duality and the second to last is a consequence of the local-to-global spectral sequence and the fact that R^1f_*F (and thus $\underline{\mathrm{Ext}}_Y^4(R^1f_*F, \mathcal{O}_Y)$) is supported in dimension zero.

If $p = -1$ this is enough. For $p = 0$ we also need to check that $\mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{D}(F), \mathcal{C}) = 0$.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{D}(F), \mathcal{C}) &= \mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{D}(F), \mathbb{D}(\mathcal{C})) \\ &= \mathrm{Hom}_{\mathcal{D}(X) \circ p} (F, \mathcal{C}) \\ &= \mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{C}, F) = 0\end{aligned}$$

where the last identity follows from $F \in {}^0\mathcal{F}$.

On the other hand, let $T \in {}^p\mathcal{T}_\bullet$. We need to check that $f_*\mathbb{D}(T) = 0$.

$$\begin{aligned}f_*\mathbb{D}(T) &= H^0(Rf_*R\mathbf{Hom}_X(T, \mathcal{O}_X)[2]) \\ &= H^2(R\mathbf{Hom}_Y(f_*T, \mathcal{O}_X)) \\ &= \underline{\mathrm{Ext}}_Y^2(f_*T, \mathcal{O}_X) \\ &= \mathrm{Ext}_Y^2(f_*T, \mathcal{O}_X) = 0\end{aligned}$$

where the last two equalities follow again from Serre duality and the dimension of the support of f_*T .

For $p = 0$ it is enough. For $p = -1$ we also need to check that $\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{C}, \mathbb{D}(T)) = 0$.

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{C}, \mathbb{D}(T)) &= \mathrm{Hom}_{\mathcal{D}(X)}(\mathbb{D}(\mathcal{C}), \mathbb{D}(T)) \\ &= \mathrm{Hom}_{\mathcal{D}(X)}(T, \mathcal{C}) = 0,\end{aligned}$$

which concludes the proof. ■

We now want to apply the duality functor, or better $\mathbb{D}' = \mathbb{D}[1]$, to our Hall algebras. As the category ${}^p\mathcal{F}[1]$ (resp. ${}^p\mathcal{T}_\bullet$) is closed by extensions we have an algebra $H_\infty({}^p\mathcal{F}[1])$ (resp. $H_\infty({}^p\mathcal{T}_\bullet)$) spanned by morphisms $[W \rightarrow {}^p\mathfrak{F}[1]]$ (resp. $[W \rightarrow {}^q\mathfrak{T}_\bullet]$). Notice that while the first is a subalgebra of $H_\infty({}^p\mathcal{A})$, the second is a

subalgebra of both $H_\infty({}^p\mathcal{A})$ and $H_\infty(\mathcal{A})$ as a distinguished triangle with vertices lying in ${}^p\mathcal{T}_\bullet$ is an exact sequence in both ${}^p\mathcal{A}$ and \mathcal{A} .

3.12 Proposition. The functor \mathbb{D}' induces an anti-isomorphism between $H_\infty({}^q\mathcal{T}_\bullet)$ and $H_\infty({}^p\mathcal{F}[1])$. Furthermore the following identities hold.

$$\begin{aligned}\mathbb{D}'(1_{{}^q\mathcal{T}_\bullet}) &= 1_{{}^p\mathcal{F}[1]} \\ \mathbb{D}'(1_{{}^q\mathcal{T}_\bullet}^\mathcal{O}) &= 1_{{}^p\mathcal{F}[1]}^\mathcal{O}\end{aligned}$$

Proof: Duality \mathbb{D}' induces an isomorphism between stacks ${}^q\mathcal{T}_\bullet$ and ${}^p\mathcal{F}[1]$. The anti-isomorphism between the Hall algebras is then defined by taking a class $[W \rightarrow {}^q\mathcal{T}_\bullet]$ to $[W \rightarrow {}^q\mathcal{T}_\bullet \rightarrow {}^p\mathcal{F}[1]]$ and noticing that duality flips extensions [Bri10a, Section 5.4]. Clearly this takes the element $1_{{}^q\mathcal{T}_\bullet}$ to $1_{{}^p\mathcal{F}[1]}$, while the second identity requires a bit of work.

Two remarks are in order. The first is that given any $T \in {}^q\mathcal{T}_\bullet$,

$$\mathrm{Hom}_X(\mathcal{O}_X, T) = \mathrm{Hom}_X(\mathbb{D}'(T), \mathcal{O}_X[3]) = \mathrm{Hom}_X(\mathcal{O}_X, \mathbb{D}'(T))^\vee.$$

The second is that, if $T \in {}^q\mathcal{T}_\bullet$ and $F \in {}^p\mathcal{F}$, then $\dim_{\mathbb{C}} H^0(X, T) = \chi(T)$ and similarly $\dim_{\mathbb{C}} H^1(X, F) = -\chi(F)$. This is useful since, for a family of coherent sheaves, the Euler characteristic is locally constant on the base. Thus we can decompose the stack ${}^q\mathcal{T}_\bullet$ as a disjoint union according to the value of the Euler characteristic. We have a corresponding decomposition of ${}^q\mathcal{T}_\bullet^\mathcal{O}$ and we write ${}^q\mathcal{T}_{\bullet, n}^\mathcal{O}$ for the n th component of this disjoint union. This space maps down to ${}^q\mathcal{T}_{\bullet, n}$ by forgetting the section. Similarly, the space $\mathbb{A}^n \times {}^q\mathcal{T}_{\bullet, n}$ projects onto ${}^q\mathcal{T}_{\bullet, n}$. As these two maps are Zariski fibrations with same fibres the stacks ${}^q\mathcal{T}_{\bullet, n}^\mathcal{O}$ and $\mathbb{A}^n \times {}^q\mathcal{T}_{\bullet, n}$ represent the same element in the Grothendieck ring. This argument is then extended to the whole ${}^q\mathcal{T}_{\bullet, n}^\mathcal{O}$ proving that

$$[{}^q\mathcal{T}_\bullet^\mathcal{O}] = \left[\bigsqcup_n \mathbb{A}^n \times {}^q\mathcal{T}_{\bullet, n} \right].$$

We can proceed analogously for ${}^p\mathcal{F}[1]$. The component ${}^p\mathcal{F}[1]_n^\mathcal{O}$ represents the same element as $\mathbb{A}^n \times {}^p\mathcal{F}[1]_n$. The first remark above implies that duality \mathbb{D}' takes ${}^q\mathcal{T}_{\bullet, n}^\mathcal{O}$ to ${}^p\mathcal{F}[1]_n$, which lets us conclude. ■

Thus in our infinite-type Hall algebra we deduce that $1_{{}^p\mathcal{F}[1]}^\mathcal{O} = \mathbb{D}'(1_{{}^q\mathcal{T}_\bullet}^\mathcal{O}) = \mathbb{D}'(\mathcal{H}_\bullet^\# * 1_{{}^q\mathcal{T}_\bullet}) = 1_{{}^p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#)$. Accordingly, we have the following identities.

$$1_{{}^p\mathcal{F}[1]}^\mathcal{O} = 1_{{}^p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#)$$

and

$${}^p\mathcal{H} * 1_{{}^p\mathcal{F}[1]} = 1_{{}^p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#) * \mathcal{H}.$$

3.6. Laurent Elements. It becomes important at this point to restrict our attention to $D_{\leq 1}^b(X)$, the bounded derived category of coherent sheaves supported in dimensions at most one. The constructions we have done so far restrict immediately by appending a ‘ ≤ 1 ’ subscript, essentially due to the fact that if two complexes have support contained in a subset S then any extension will have support contained in S as well. Notice that ${}^p\mathcal{F}_{\leq 1} = {}^p\mathcal{F}$.

Our objective is to get rid of the spurious $1_{{}^p\mathcal{F}[1]}$ ’s in the identity above. This is achieved by constructing a (weak) stability condition on ${}^p\mathcal{A}_{\leq 1}$ (in the sense of [JS08, Definition 3.5]), with values in the ordered set $\{0, 1, 2\}$, such that ${}^p\mathcal{F}[1]$ manifests as the class of semi-stable objects of $\mu = 2$. Before we do that, however, we want to define a sort of completed Hall algebra $H({}^p\mathcal{A})_\Lambda$ (analogous to what is done in [Bri10a, Section 5.2]) which morally sits in between $H({}^p\mathcal{A}_{\leq 1})$ and $H_\infty({}^p\mathcal{A}_{\leq 1})$. The reason we need to do so is simple. On one hand the Hall algebra constructed in the previous section only includes spaces that are of finite type, on the other the infinite type Hall algebra is much too big to support an integration morphism. To deal with objects such as the Hilbert scheme of curves and points of X we allow our spaces to be *locally* of finite type while imposing a sort of ‘Laurent’ property.

We previously mentioned that $H({}^p\mathcal{A})$ is graded by $N(X)$. There is a subgroup $N_{\leq 1}(X)$ generated by sheaves supported in dimension at most one and $H({}^p\mathcal{A}_{\leq 1})$ is graded by it. We also notice [Bri10a, Lemma 2.2] that the Chern character induces an isomorphism

$$N_{\leq 1}(X) \ni [E] \longmapsto (\mathrm{ch}_2 E, \mathrm{ch}_3 E) \in N_1(X) \oplus N_0(X)$$

where by $N_1(X)$ we mean the group of curve-classes modulo numerical equivalence, and $N_0(X) \simeq \mathbb{Z}$. Henceforth we tacitly identify $N_{\leq 1}(X)$ with $N_1(X) \oplus \mathbb{Z}$.

We have a pushforward morphism

$$f_* : N_1(X) \longrightarrow N_1(Y)$$

which is surjective and has kernel $N_1(X/Y)$. Thus we have an exact sequence $N_1(X/Y) \hookrightarrow N_1(X) \rightarrow N_1(Y)$ of free abelian groups (of finite rank) so $N_1(X) \cong N_1(Y) \oplus N_1(X/Y)$. Elements of $N_{\leq 1}(X)$ can then be described by triples $(\gamma, \delta, n) \in N_1(Y) \oplus N_1(X/Y) \oplus \mathbb{Z}$. We denote the image of ${}^p\mathcal{A}_{\leq 1}$ (via the Chern character) in $N_{\leq 1}(X)$ by ${}^p\Delta$ (this is the cone of perverse coherent sheaves supported in dimensions ≤ 1). The algebra $H({}^p\mathcal{A}_{\leq 1})$ is graded by ${}^p\Delta$. Finally, by $\mathcal{E} \subset N_1(X/Y)$ we denote the *effective* curve classes in X which are contracted by f .

3.13 Definition. We define a subset $L \subset {}^p\Delta$ to be *Laurent* if the following conditions hold:

- for all γ there exists an $n(\gamma, L)$ such that for all δ, n with $(\gamma, \delta, n) \in L$ one has that $n \geq n(\gamma, L)$;
- for all γ, n there exists a $\delta(\gamma, n, L) \in \mathcal{E}$ such that for all δ with $(\gamma, \delta, n) \in L$ one has that $\delta \leq \delta(\gamma, n, L)$.¹¹

We denote by Λ the set of all Laurent subsets of ${}^p\Delta$.

We have the following lemma.

3.14 Lemma. The set Λ of Laurent subsets of ${}^p\Delta$ satisfies the two following properties.

- (1) If $L_1, L_2 \in \Lambda$ then $L_1 + L_2 \in \Lambda$.
- (2) If $\alpha \in {}^p\Delta$ and $L_1, L_2 \in \Lambda$ then there exist only finitely many decompositions $\alpha = \alpha_1 + \alpha_2$ with $\alpha_j \in L_j$.

Proof: We start by proving (1). Fix a γ and let $(\gamma, \delta, n) \in L_1 + L_2$. By [KM98, Corollary 1.19] there are only finitely many decompositions $\gamma = \gamma_1 + \gamma_2$ with $\gamma_i \geq 0$ (i.e. with γ_i effective). Given a decomposition $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$, with $(\gamma_i, \delta_i, n_i) \in L_i$, we know that $n_i \geq n(\gamma_i, L_i)$ so $n = n_1 + n_2 \geq n(\gamma_1, L_1) + n(\gamma_2, L_2)$. By letting the γ_i 's vary we obtain the desired lower bound for n .

Fix now γ, n . We want to find a bound for the possible δ 's such that $(\gamma, \delta, n) \in L_1 + L_2$. By the argument above we know that for decompositions $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$ with $(\gamma_i, \delta_i, n_i) \in L_i$ the possible combinations of γ_i and n_i are finite. Fix such a decomposition $(\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$. We know that $\delta_i \leq \delta(\gamma_i, n_i, L_i)$. Thus $\delta = \delta_1 + \delta_2 \leq \delta(\gamma_1, n_1, L_1) + \delta(\gamma_2, n_2, L_2)$. Take now another decomposition $(\gamma'_1 + \gamma'_2, \delta'_1 + \delta'_2, n'_1 + n'_2)$. Running the same argument we have that $\delta \leq \delta(\gamma'_1, n'_1, L_1) + \delta(\gamma'_2, n'_2, L_2)$. Finally, as $\delta(\gamma_i, n_i, L_i), \delta(\gamma'_i, n'_i, L_i) \geq 0$, we conclude $\delta \leq \sum_i \delta(\gamma_i, n_i, L_i) + \delta(\gamma'_i, n'_i, L_i)$. By taking the sum for all possible decompositions we have our upper bound for δ .

Let us now prove (2). Fix a class $\alpha = (\gamma, \delta, n) \in {}^p\Delta$ and two Laurent subsets L_1, L_2 . Again by [KM98, Corollary 1.19] we know that the possible decompositions $\gamma = \gamma_1 + \gamma_2$ are finitely many. Thus we may fix γ_1 and γ_2 . Given a decomposition $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$, the possible values occurring in n_1, n_2 are again finitely many as $n_i \geq n(\gamma_i, L_i)$. Thus we may take n_1, n_2 also to be fixed. Finally, the combinations $(\gamma, \delta, n) = (\gamma_1 + \gamma_2, \delta_1 + \delta_2, n_1 + n_2)$ are again a finite number, as $\delta = \delta_1 + \delta_2$ lives in $\delta(\gamma_1, n_1, L_1) + \delta(\gamma_2, n_2, L_2) - \mathcal{E}$ (thus we can apply [KM98, Corollary 1.19] again). ■

We now have all the ingredients to define a Λ -completion $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ of $H({}^p\mathcal{A}_{\leq 1})$. Let us give a general definition.

3.15 Definition. Let R be a ${}^p\Delta$ -graded associative \mathbb{Q} -algebra. We define R_Λ to be the vector space of formal series

$$\sum_{(\gamma, \delta, n)} x_{(\gamma, \delta, n)}$$

with $x_{(\gamma, \delta, n)} \in R_{(\gamma, \delta, n)}$ and $x_{(\gamma, \delta, n)} = 0$ outside a Laurent subset. We equip this vector space with a product

$$x \cdot y = \sum_{\alpha \in {}^p\Delta} \sum_{\alpha_1 + \alpha_2 = \alpha} x_{\alpha_1} \cdot y_{\alpha_2}.$$

The algebra R is included in R_Λ as any finite set is Laurent. To a morphism $R \rightarrow S$ of ${}^p\Delta$ -graded algebras corresponds an obvious morphism $R_\Lambda \rightarrow S_\Lambda$.

There is a subalgebra

$$\mathbb{Q}_o[{}^p\Delta] \subset \mathbb{Q}_o[{}^p\Gamma]$$

¹¹For $\delta, \delta' \in N_1(X/Y)$, by the notation $\delta \leq \delta'$ we mean $\delta' - \delta \in \mathcal{E}$ or equivalently $\delta - \delta' \in -\mathcal{E}$. In general we will write $\alpha \geq 0$ to denote that a certain class is effective.

spanned by symbols q^α with $\alpha \in {}^p\Delta$. Notice that the Poisson structure on $\mathcal{Q}_\sigma[{}^p\Delta]$ is trivial as the Euler form on $N_{\leq 1}(X)$ is identically zero. The integration morphism restricts to $I : H_{\text{sc}}({}^p\mathcal{A}_{\leq 1}) \rightarrow \mathcal{Q}_\sigma[{}^p\Delta]$ and so, by taking Λ -completions, we have a morphism

$$I_\Lambda : H_{\text{sc}}({}^p\mathcal{A}_{\leq 1})_\Lambda \longrightarrow \mathcal{Q}_\sigma[{}^p\Delta]_\Lambda.$$

3.16 Remark. Notice that given an algebra R as above and an element $r \in R$ with $r_{(0,0,0)} = 0$, the element $1 - r$ is invertible in R_Λ . This is due to the fact that the series

$$\sum_{k \geq 0} r^k$$

makes sense in R_Λ .

Now it's time to have a look at what the elements of $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ look like. Let \mathfrak{M} be an algebraic stack locally of finite type over \mathbb{C} mapping down to ${}^p\mathfrak{A}_{\leq 1}$ and denote by \mathfrak{M}_α the preimage under ${}^p\mathfrak{A}_\alpha$, for $\alpha \in {}^p\Delta$. We say that

$$[\mathfrak{M} \rightarrow {}^p\mathfrak{A}_{\leq 1}] \in H_\infty({}^p\mathcal{A}_{\leq 1})$$

is *Laurent* if \mathfrak{M}_α is a stack of finite type for all $\alpha \in {}^p\Delta$ and if \mathfrak{M}_α is empty for α outside a Laurent subset. Such a Laurent element gives an element of $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ by considering $\sum_\alpha \mathfrak{M}_\alpha$. The algebra $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ is spanned by these Laurent elements.

3.17 Proposition. The element $1_{{}^p\mathcal{F}[1]}$ is Laurent.

Proof: Let $F \in {}^p\mathcal{F}$ and let (γ, δ, n) be the class in $N_{\leq 1}(X)$ corresponding to $[F[1]] = -[F]$. By [SGA6, Proposition X-1.1.2] we know that in the Grothendieck group F decomposes as

$$F = \sum_i l_i [\mathcal{O}_{C_i}] + \tau$$

where the C_i are the curves comprising the irreducible components of the support of F (which is contained in the exceptional locus of f), where $l_i \geq 0$ and where τ is supported in dimension zero. From this decomposition we infer that $\gamma = 0$ and $\delta \leq 0$. Finally, Riemann-Roch tells us that n is minus the Euler characteristic of F and Lemma 1.3 gives us that $n \geq 0$. ■

Notice also that by the remark above $1_{{}^p\mathcal{F}[1]}$ is invertible in $H({}^p\mathcal{A}_{\leq 1})_\Lambda$. Similarly $1_{{}^p\mathcal{F}[1]}$ belongs to (and is invertible in) $H({}^p\mathcal{A}_{\leq 1})_\Lambda$.

3.18 Proposition. The element ${}^p\mathcal{H}_{\leq 1}$ is Laurent.

Proof: By [Bri02, Theorem 5.5] if we fix a numerical class $\alpha \in N_{\leq 1}(X)$ then the space ${}^p\text{Hilb}_{X/Y, \alpha}$ is of finite type (it is in fact a projective scheme). Thus we are left with checking the second half of the Laurent property. Notice that if P is a perverse coherent sheaf with $\dim \text{supp } P \leq 1$ corresponding to a numerical class (γ, δ, n) then $n \geq 0$ (this follows from Lemma 1.3). Moreover, we only really need to focus on exact sequences of both coherent and perverse sheaves, that is on points of $\text{Hilb}_X \cap {}^p\text{Hilb}_{X/Y, \leq 1}$ (which we temporarily denote by Pilb_X). This is a consequence of the fact that given a quotient $\mathcal{O}_X \twoheadrightarrow P$ in ${}^p\mathcal{A}$, with P of class (γ, δ, n) , we can consider the torsion torsion-free exact sequence

$$F[1] \hookrightarrow P \twoheadrightarrow T.$$

In fact, $F[1]$ does not contribute towards γ , contributes negatively towards δ and positively towards n , as seen in the previous proposition. Thus we just need to study the possible classes of T . Finally, $\mathcal{O}_X \twoheadrightarrow P \twoheadrightarrow T$ is a quotient in ${}^p\mathcal{A}$ but glancing at the cohomology sheaves long exact sequence tells us that it is indeed a quotient in \mathcal{A} as well. By what we remarked at the beginning of this proof, we know that T contributes positively towards n as well. Thus we only need to check that, having chosen a γ and an n , there exists an upper bound δ_0 such that $\text{Pilb}_{X, (\gamma, \delta, n)}$ is empty for $\delta \geq \delta_0$.

Notice that the pushforward induces a morphism from ${}^p\text{Hilb}_{X/Y}$ to Hilb_Y . We consider its restriction to Pilb_X . We would like for the pullback functor to induce a morphism going in the opposite direction. A flat family of sheaves on Y might, however, cease to be flat once pulled back on X . To remedy we impose this condition by hand. We define a subfunctor Filb_Y of Hilb_Y by the rule

$$\text{Filb}_Y(S) = \left\{ \mathcal{O}_{Y_S} \twoheadrightarrow G \mid G, f_S^* G \text{ flat over } S \right\}.$$

If \mathcal{U} is the structure sheaf of the universal subscheme for Hilb_Y on $Y \times \text{Hilb}_Y$ then one can see that Filb_Y is represented by the flattening stratification of Hilb_Y with respect to $f_{\text{Hilb}_Y}^* \mathcal{U}$. From this we deduce that if we fix a numerical class (γ, n) on Y then $\text{Filb}_{Y,(\gamma,n)}$ is of finite type.

We claim that the composition of pushing forward and pulling up as just described, $\text{Pilb}_X \rightarrow \text{Filb}_Y \rightarrow \text{Pilb}_X$, is the identity. Let us see first why this is true on geometric points. Take an exact sequence of both coherent and perverse sheaves

$$I \hookrightarrow \mathcal{O}_X \twoheadrightarrow E.$$

Applying the counit of the adjunction $f^* \dashv f_*$ (and using the fact that the objects above are both sheaves and perverse sheaves) we obtain a commutative diagram

$$\begin{array}{ccccccc} f^* f_* I & \longrightarrow & \mathcal{O}_X & \longrightarrow & f^* f_* E & \longrightarrow & 0 \\ \downarrow & & \downarrow \text{id} & & \downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & \mathcal{O}_X & \longrightarrow & E \longrightarrow 0 \end{array}$$

with exact rows. By [Bri02, Proposition 5.1] we have that $f^* f_* I \rightarrow I$ is surjective and so, by a simple diagram chase, $f^* f_* E \rightarrow f^* f_* E$ is an isomorphism. This argument indeed works in families, as surjectivity can be checked fibrewise.

Finally, let us fix a γ and an n and let $\text{Pilb}_{X,\gamma,n}$ be the subspace of Pilb_X where we've fixed γ and n but we let δ vary. By the previous arguments we know that $\text{Pilb}_{X,\gamma,n} \rightarrow \text{Filb}_{Y,(\gamma,n)} \rightarrow \text{Pilb}_{X,\gamma,n}$ composes to the identity. As the retract of a quasi-compact space is quasi-compact¹² we obtain that $\text{Pilb}_{X,\gamma,n}$ is of finite type, which is enough to conclude. ■

3.19 Proposition. The element $\mathcal{H}_{\leq 1}$ is Laurent.

Proof: It is a known fact that for a fixed numerical class $\alpha \in N_{\leq 1}(X)$ the scheme $\text{Hilb}_{X,\alpha}$ is of finite type (it is in fact a projective scheme). To prove the second half of the Laurent property we first remark that the structure sheaf of a curve in X has positive Euler characteristic. Hence we only need to find a bound for δ once we fix γ and n . We start off with the identity

$${}^p \mathcal{H}_{\leq 1} * 1_{{}^p \mathcal{F}[1]} = 1_{{}^p \mathcal{F}[1]} * \mathcal{H}_{\leq 1}$$

in $H_\infty({}^p \mathcal{A}_{\leq 1})$. By directly applying our definition of $*$ we see that the left hand side is represented by a morphism $[W \rightarrow {}^p \mathfrak{A}_{\leq 1}]$, given by the top row of the following diagram.

$$\begin{array}{ccccc} W & \longrightarrow & {}^p \mathfrak{A}_{\leq 1}^{(2)} & \xrightarrow{b} & {}^p \mathfrak{A}_{\leq 1} \\ \downarrow & & \downarrow (a_1, a_2) & & \\ {}^p \mathfrak{F}[1]^{\mathcal{O}} \times \text{Hilb}_{X,\leq 1} & \longrightarrow & {}^p \mathfrak{A}_{\leq 1} \times {}^p \mathfrak{A}_{\leq 1} & & \end{array}$$

Similarly, the right hand side is represented by a morphism $[Z \rightarrow {}^p \mathfrak{A}_{\leq 1}]$. The main tool we use for the proof is the cover $\{{}^p \mathfrak{A}_\alpha\}_\alpha$ of ${}^p \mathfrak{A}_{\leq 1}$, with $\alpha \in {}^p \Delta$ ranging inside the effective cone of perverse coherent sheaves.

By taking preimages through b we obtain an open cover $\{U_\alpha\}_\alpha$ of ${}^p \mathfrak{A}_{\leq 1}^{(2)}$. Concretely, U_α parameterises exact sequences $P_1 \hookrightarrow P \twoheadrightarrow P_2$ in ${}^p \mathcal{A}_{\leq 1}$ with P of class α .

On the other hand, we can cover ${}^p \mathfrak{A}_{\leq 1} \times {}^p \mathfrak{A}_{\leq 1}$ by taking products ${}^p \mathfrak{A}_{\alpha_1} \times {}^p \mathfrak{A}_{\alpha_2}$. By pulling back via (a_1, a_2) we produce an open cover $\{U_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2}$ of ${}^p \mathfrak{A}_{\leq 1}^{(2)}$. The space U_{α_1, α_2} parameterises exact sequences $P_1 \hookrightarrow P \twoheadrightarrow P_2$ in ${}^p \mathcal{A}_{\leq 1}$ with P_1 of class α_1 and P_2 of class α_2 . Notice that the collection $\{U_{\alpha_1, \alpha_2}\}_{\alpha_1 + \alpha_2 = \alpha}$ is an open cover of U_α .

By pulling back these covers of ${}^p \mathfrak{A}_{\leq 1}^{(2)}$ we obtain open covers $\{W_\alpha\}_\alpha$ and $\{W_{\alpha_1, \alpha_2}\}_{\alpha_1, \alpha_2}$ of W . The same can be done for Z .

We remind ourselves that we think of a class α as a triple (γ, δ, n) . If we fix a γ and an n , it is a consequence of ${}^p \mathcal{H}_{\leq 1} * 1_{{}^p \mathcal{F}[1]}$ being Laurent that there exists an δ' such that $Z_{(\gamma, \delta, n)} = \emptyset$ for $\delta \geq \delta'$. Because of the identity above, the same holds for $W_{(\gamma, \delta, n)}$.

What we need to prove is that, once we fix γ and n_2 , the space $\text{Hilb}_{X,(\gamma, \delta_2, n_2)}$ is empty for large δ_2 . Fix δ_1, n_1 such that ${}^p \mathfrak{F}[1]_{(0, \delta_1, n_1)}^{\mathcal{O}} \neq \emptyset$. The space representing the product

$$1_{{}^p \mathfrak{F}[1]_{(0, \delta_1, n_1)}^{\mathcal{O}}} * \text{Hilb}_{X,(\gamma, \delta_2, n_2)}$$

¹²If $A \rightarrow B \rightarrow A$ composes to the identity, one can start with an open cover $\{A_i\}$ and pull it back to a cover $\{B_i\}$ of B . Pick a finite subcover $\{B_j\}$ and pull it back to A . This is a finite subcover of $\{A_i\}$.

is $W_{(0,\delta_1,n_1),(\gamma,\delta_2,n_2)} \subset W_{(\gamma,\delta_1+\delta_2,n_1+n_2)}$. We have already remarked that for fixed γ, n_1, n_2 we have an upper bound δ' such that $W_{(\gamma,\delta_1+\delta_2,n_1+n_2)} = \emptyset$ for $\delta_1 + \delta_2 \geq \delta'$. As ${}^p\mathfrak{F}[1]_{(0,\delta_1,n_1)}^\circ \neq \emptyset$, we conclude that $\text{Hilb}_{X,(\gamma,\delta_2,n_2)} = \emptyset$ for $\delta_2 \geq \delta' - \delta_1$, in particular the same is true for $\delta_2 \geq 0$. \blacksquare

We need to interpret Proposition 3.12 in the Laurent setting. Duality \mathbb{D}' acts on $N_{\leq 1}(X)$ by taking a class (γ, δ, n) to $(-\gamma, -\delta, n)$. Even more concretely, an element $T \in {}^q\mathcal{T}_\bullet$ of class $(0, \delta, n)$ is sent to an element $\mathbb{D}'(T) \in {}^p\mathcal{F}$ of class $(0, -\delta, n)$. Therefore we can construct an algebra $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$ by defining Laurent subsets with the opposite sign conventions for δ . The elements $1_{{}^q\mathcal{T}_\bullet}$ and $1_{{}^q\mathcal{T}_\bullet}^\circ$ belong to this algebra $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$ by Propositions 3.12 and 3.17. The element $\mathcal{H}_\bullet^\#$ also belongs to $H({}^q\mathcal{T}_\bullet)_{\Lambda'}$ by running an argument similar to the proof above.

Hence, duality \mathbb{D}' defines a morphism $H({}^q\mathcal{T}_\bullet)_{\Lambda'} \rightarrow H({}^p\mathcal{F}[1])_\Lambda$ and Proposition 3.12 remains valid in this context. In particular, using what we observed earlier about the action of \mathbb{D}' on $N_{\leq 1}(X)$, we have an identity

$$(3.20) \quad I_\Lambda(\mathbb{D}'(\mathcal{H}_\bullet^\#)) = \text{PT}_f^\vee := \sum_{\substack{(\beta, n) \in N_{\leq 1}(X) \\ f_*\beta=0}} (-1)^n \text{PT}_X(-\beta, n) q^{(\beta, n)}.$$

Going back to $H({}^p\mathcal{A}_{\leq 1})_\Lambda$, we have an identity

$$(3.21) \quad {}^p\mathcal{H}_{\leq 1} = 1_{{}^p\mathcal{F}[1]} * \mathbb{D}'(\mathcal{H}_\bullet^\#) * \mathcal{H}_{\leq 1} * 1_{{}^p\mathcal{F}[1]}^{-1}.$$

What keeps us from simply applying the integration morphism I_Λ is that, although $\mathbb{D}'(\mathcal{H}_\bullet^\#)$ and $\mathcal{H}_{\leq 1}$ lie in $H_{\text{reg}}({}^p\mathcal{A}_{\leq 1})_\Lambda$, $1_{{}^p\mathcal{F}[1]}$ does not. We want to proceed analogously as in [Bri10a, Section 6.3], proving that

$$I_\Lambda({}^p\mathcal{H}_{\leq 1}) = I_\Lambda(\mathbb{D}'(\mathcal{H}_\bullet^\#)) \cdot I_\Lambda(\mathcal{H}_{\leq 1})$$

holds nevertheless.

3.7. A Stability Condition. We now make our way to define a (weak) stability condition (in the sense of [JS08, Definition 3.5]) on ${}^p\mathcal{A}_{\leq 1}$. In our set-up such a stability condition is simply a function μ

$$\mu : {}^p\Delta \rightarrow \{0, 1, 2\}$$

from the numerical effective cone to the ordered set $\{0, 1, 2\}$ satisfying the (weak) see-saw property. We define the stability condition μ as follows:

$$(\gamma, \delta, n) \mapsto \begin{cases} 0 & \text{if } \gamma > 0 \\ 1 & \text{if } \gamma = 0, \delta \geq 0 \\ 2 & \text{if } \gamma = 0, \delta < 0. \end{cases}$$

Recall that an object P is said to be *semistable* if for all proper subobjects (in ${}^p\mathcal{A}$) $P' \subset P$ we have $\mu(P') \leq \mu(P/P')$.

3.22 Proposition. Any μ -semistable object belongs to either ${}^p\mathcal{F}[1]$ or ${}^p\mathcal{T}_{\leq 1}$. The μ -semistable objects satisfying $\mu = 2$ consist precisely of the elements of ${}^p\mathcal{F}[1]$.

Proof: Let P be any semistable perverse coherent sheaf. Consider the torsion torsion-free exact sequence

$$F[1] \hookrightarrow P \twoheadrightarrow T.$$

If $F[1] \neq 0$ and $T \neq 0$ then, by semistability, $2 = \mu(F[1]) \leq \mu(T) \leq 1$ which is impossible. Thus a semistable object must be either torsion or torsion-free.

On the other hand, let $F[1] \in {}^p\mathcal{F}[1]$. By the definition $\mu(F[1]) = 2$. As quotients of $F[1]$ lie in ${}^p\mathcal{F}[1]$ (and thus have $\mu = 2$) we conclude. \blacksquare

3.23 Proposition. In $H({}^p\mathcal{A}_{\leq 1})_\Lambda$ we have $1_{{}^p\mathcal{F}[1]} = \exp \epsilon$, with $\eta = \epsilon \cdot [\mathbb{G}_m] \in H_{\text{reg}}({}^p\mathcal{A}_{\leq 1})_\Lambda$ a regular element. Furthermore the automorphism

$$\text{Ad}_{1_{{}^p\mathcal{F}[1]}} : H({}^p\mathcal{A}_{\leq 1})_\Lambda \longrightarrow H({}^p\mathcal{A}_{\leq 1})_\Lambda$$

preserves the regular elements. The induced Poisson automorphism of $H_{\text{sc}}({}^p\mathcal{A}_{\leq 1})_\Lambda$ is given by

$$\text{Ad}_{1_{{}^p\mathcal{F}[1]}} = \exp\{\eta, -\}.$$

Proof: This is analogous to Theorem 6.3 and Corollary 6.4 in [Bri10a]. We only need to show that the stability condition μ is *permissible* in the sense of [Joy, Definition 4.7], which is the content of the following lemma. ■

3.24 Lemma. The stability condition μ is permissible.

Proof: The first fact we check is that the category ${}^p\mathcal{A}$ is noetherian. We give a direct proof for $p = -1$ and use the equivalence Φ to pass to the other side of the flop and conclude the same for $p = 0$. Let $P \in {}^p\mathcal{A}$ and let

$$P_1 \hookrightarrow P_2 \hookrightarrow \dots \hookrightarrow P$$

be an ascending chain of subobjects of P . Taking cohomologies, one has an ascending chain

$$H^{-1}(P_1) \hookrightarrow H^{-1}(P_2) \hookrightarrow \dots \hookrightarrow H^{-1}(P)$$

of coherent sheaves. As \mathcal{A} is noetherian, we may assume $H^{-1}(P_1) \simeq H^{-1}(P_2)$. Taking derived pushforward Rf_* , one obtains an ascending chain

$$Rf_*P_1 \hookrightarrow Rf_*P_2 \hookrightarrow \dots \hookrightarrow Rf_*P$$

of coherent sheaves on Y . Again, this must stabilise, thus we may assume $Rf_*(P_2/P_1) = 0$. Leray's spectral sequence quickly tells us that in fact $f_*H^0(P_2/P_1) = 0$. As $p = -1$ we have that $H^0(P_2/P_1) \in {}^p\mathcal{F}$ but, as $H^0(P_2/P_1) \in {}^p\mathcal{T}$ (by default), $H^0(P_2/P_1) = 0$. In turn, this gives a chain of surjections

$$H^0(P_1) \twoheadrightarrow H^0(P_2) \twoheadrightarrow \dots$$

of coherent sheaves. This is equivalent to the ascending chain of kernels thus it must stabilise. As a result, we can assume $H^0(P_1) \simeq H^0(P_2)$ which concludes the proof that ${}^p\mathcal{A}$ is noetherian.

Now we want to check that if $P \in {}^p\mathcal{A}_{\leq 1}$ and $[P] = 0$ in $N_{\leq 1}(X)$ then $P = 0$. By pushing forward via f we have that $[Rf_*P] = 0$ and as $Rf_*P \in \text{Coh } Y$ it follows that $Rf_*P = 0$. Now, from Leray's spectral sequence we obtain that $f_*H^{-1}(P) = f_*H^0(P) = 0$. Thus, if $p = -1$, $P = H^{-1}(P)[1]$ and, if $p = 0$, $P = H^0(P)$. In either case we reduce to dealing with a coherent sheaf and so $P = 0$.

Let now ${}^p\mathcal{A}_\alpha(i)$ be the subset of ${}^p\mathfrak{A}_{\leq 1}(\mathbb{C})$ consisting of perverse coherent sheaves which are of numerical class α and semistable with $\mu = i$. We now check that these subsets are constructible.

For $\mu = 2$, Proposition 3.22 tells us that ${}^p\mathcal{A}_\alpha(2) = {}^p\mathfrak{F}[1]_\alpha(\mathbb{C})$, which is constructible as ${}^p\mathfrak{F}[1]_\alpha$ is open in ${}^p\mathfrak{A}_{\leq 1}$.

For $\mu = 1$, we know from Proposition 3.22 that a semistable object of $\mu = 1$ lies in ${}^p\mathcal{T}$ and satisfies $\gamma = 0$. The converse is also true as the category ${}^p\mathcal{T}$ is stable under subobjects. Thus ${}^p\mathcal{A}_\alpha(1)$ is also constructible.

For $\mu = 0$ one argues similarly to the $\mu = 1$ case, only now an object in ${}^p\mathcal{T}_{\leq 1}$ with $\gamma > 0$ is semistable if and only if it does not contain any subobjects $T' \hookrightarrow T$ with $\gamma' = 0$. The corresponding subset is constructible for the following reason. Consider ${}^p\mathfrak{T}_{\gamma=0}$, the substack of ${}^p\mathfrak{T}$ parameterising objects of ${}^p\mathcal{T}$ with $\gamma = 0$. Recall the morphisms $(a_1, a_2) : {}^p\mathfrak{A}_{\leq 1}^{(2)} \rightarrow {}^p\mathfrak{A}_{\leq 1} \times {}^p\mathfrak{A}_{\leq 1}$ and $b : {}^p\mathfrak{A}_{\leq 1}^{(2)} \rightarrow {}^p\mathfrak{A}_{\leq 1}$ defining the product in $H_\infty({}^p\mathcal{A}_{\leq 1})$. We take the inclusion ${}^p\mathfrak{T}_{\gamma=0} \times {}^p\mathfrak{A}_{\leq 1} \rightarrow {}^p\mathfrak{A}_{\leq 1} \times {}^p\mathfrak{A}_{\leq 1}$, pull it back via (a_1, a_2) and take its image in ${}^p\mathfrak{A}_{\leq 1}$ through b . This defines a constructible subset of ${}^p\mathfrak{A}_{\leq 1}(\mathbb{C})$. The complement of it inside ${}^p\mathfrak{T}_{(0,\delta,n)}(\mathbb{C})$ is precisely ${}^p\mathcal{A}_{(0,\delta,n)}(0)$.

To finish, we show that μ is artinian. Consider a chain of subobjects

$$\dots \hookrightarrow P_2 \hookrightarrow P_1$$

with $\mu(P_{n+1}) \geq \mu(P_n/P_{n+1})$. Let $P' \hookrightarrow P$ be any two consecutive elements in the chain above and let Q be the quotient P'/P so that we have an exact sequence

$$P' \hookrightarrow P \twoheadrightarrow Q$$

with $\mu(P') \geq \mu(Q)$, which corresponds to the relation $(\gamma', \delta', n') + (\gamma_q, \delta_q, n_q) = (\gamma, \delta, n)$ in ${}^p\Delta$. As all the γ 's involved are effective we can assume (by going further down the chain) $\gamma' = \gamma$ so that $\gamma_q = 0$. But as $\mu(P') \geq \mu(Q)$ we have that $\gamma = \gamma' = 0$. So P', P, Q are all supported on the exceptional locus.

From this we deduce that all the n 's are positive. As $0 \leq n' \leq n$ we can assume $n' = n$, which implies $n_q = 0$. This implies that $Q \in {}^p\mathcal{F}[1]$, hence $\mu(Q) = 2$ which implies $\mu(P') = \mu(P) = 2$ and so $\delta, \delta' < 0$.

Finally, as $\delta' \geq \delta$, we can assume $\delta' = \delta$ and so $\delta_q = 0$, from which we gather that Q is the shift of a skyscraper sheaf. As sheaves in ${}^p\mathcal{F}$ have no sections we have that $Q = 0$, which concludes the proof. ■

3.8. Conclusion. At last, we have all the ingredients to prove our main result. Recall that the Poisson bracket on $\mathbb{Q}_\sigma[{}^p\Delta]$ is trivial, so Proposition 3.23, together with (3.21), yields the identity

$$I_\Lambda({}^p\mathcal{H}_{\leq 1}) = I_\Lambda(\mathbb{D}'(\mathcal{H}_\bullet^\#)) \cdot I_\Lambda(\mathcal{H}_{\leq 1}).$$

By [Bri10a, Lemma 5.5] we gather that

$$I_\Lambda(\mathcal{H}_{\leq 1}) = \text{DT}_X := \sum_{\beta, n} (-1)^n \text{DT}_X(\beta, n) q^{(\beta, n)}.$$

By (3.20) we know that

$$I_\Lambda(\mathbb{D}'(\mathcal{H}_\bullet^\#)) = \text{PT}_f^\vee := \sum_{\substack{(\beta, n) \in \mathcal{N}_{\leq 1}(X) \\ f_* \beta = 0}} (-1)^n \text{PT}_X(-\beta, n) q^{(\beta, n)}.$$

Thus, if we denote $I_\Lambda({}^p\mathcal{H}_{\leq 1})$ by $\text{DT}_{X/Y}$, we obtain

$$(3.25) \quad \text{DT}_{X/Y} = \text{PT}_f^\vee \cdot \text{DT}_X$$

By [Bri10a, Theorem 1.1] one has

$$\begin{aligned} \text{DT}_{X,0} \cdot \text{PT}_f^\vee &= \text{DT}_f^\vee, \text{ where } \text{DT}_{X,0} := \sum_n (-1)^n \text{DT}_X(0, n) q^{(0, n)} \\ \text{and } \text{DT}_f^\vee &:= \sum_{\substack{(\beta, n) \in \mathcal{N}_{\leq 1}(X) \\ f_* \beta = 0}} (-1)^n \text{DT}_X(-\beta, n) q^{(\beta, n)}. \end{aligned}$$

Hence we can rewrite (3.25) solely in terms of DT invariants

$$(3.26) \quad \text{DT}_{X,0} \cdot \text{DT}_{X/Y} = \text{DT}_f^\vee \cdot \text{DT}_X.$$

We now want to understand how to pass over to the other side of the flop. The derived equivalence Φ induces an equivalence between ${}^q\mathcal{A}_{\leq 1}^+$ and ${}^p\mathcal{A}_{\leq 1}$ [Bri02, (4.8)]. In turn this yields an isomorphism between the stack ${}^q\mathfrak{A}_{\leq 1}^+$ and ${}^p\mathfrak{A}_{\leq 1}$, from which one obtains an isomorphism between $H({}^q\mathcal{A}_{\leq 1}^+)$ and $H({}^p\mathcal{A}_{\leq 1})$, which we still denote by Φ .

On the level of curve-classes we take $\phi_* : N_1(X^+) \rightarrow N_1(X)$ to be the inverse of the transpose of the strict transform of divisors. This can be extended to an isomorphism $\phi_* : \mathcal{Q}_\sigma[{}^q\Delta^+] \rightarrow \mathcal{Q}_\sigma[{}^p\Delta]$, defined by $q^{(\beta, n)} \mapsto q^{(\phi_*\beta, n)}$. From the proof of [Bri02, (4.6)] we infer that Φ is compatible with ϕ_* , in the sense that the following diagram commutes.

$$\begin{array}{ccc} H({}^q\mathcal{A}_{\leq 1}^+)_\Lambda & \xrightarrow{\Phi} & H({}^p\mathcal{A}_{\leq 1})_\Lambda \\ I_\Lambda \downarrow & & \downarrow I_\Lambda \\ \mathcal{Q}_\sigma[{}^q\Delta^+]_\Lambda & \xrightarrow{\phi_*} & \mathcal{Q}_\sigma[{}^p\Delta]_\Lambda \end{array}$$

As $\Phi(\mathcal{O}_{X^+}) = \mathcal{O}_X$, we observe that $\Phi({}^q\mathcal{H}_{\leq 1}^+) = {}^p\mathcal{H}_{\leq 1}$ and thus

$$\text{DT}_{X/Y} = \phi_* \text{DT}_{X^+/Y}$$

so that the identity

$$\text{PT}_f^\vee \cdot \text{DT}_X = \phi_* \left(\text{PT}_{f^+}^\vee \cdot \text{DT}_{X^+} \right)$$

holds.

Finally, one can express this last identity purely in terms of DT invariants as follows. The series $\text{DT}_{X,0}$ is equal to $\chi_{\text{top}}(X)M(q)$, where $\chi_{\text{top}}(X)$ is the topological Euler characteristic of X and $M(q)$ is the McMahon function [BF08]. By [Bat99] we know that $\chi_{\text{top}}(X) = \chi_{\text{top}}(X^+)$, so that $\text{DT}_{X^+,0} = \phi_* \text{DT}_{X,0}$. The combination of all these facts gives rise to our main result.

3.27 Theorem. Assume Situation 1.1. Then the identity

$$\text{DT}_f^\vee \cdot \text{DT}_X = \phi_* \left(\text{DT}_{f^+}^\vee \cdot \text{DT}_{X^+} \right)$$

holds.

APPENDIX A. SUBSTACKS

At the core of the construction of the Hall algebra of an abelian category lies the existence of a moduli stack¹³ parameterising its objects (and a moduli of short exact sequences). In our case this amounts, first of all, to proving the existence of the moduli stack ${}^p\mathfrak{A}$, parameterising perverse coherent sheaves on our variety X . We have mentioned in the first section that as the category ${}^p\mathcal{A}$ is the heart of a t-structure in the derived category $D^b(X)$, its objects have no negative self-extensions. This simple remark is actually key, as we construct ${}^p\mathfrak{A}$ as an open substack of one big moduli stack \mathfrak{Mum}_X , which Lieblich refers to as *the mother of all moduli of sheaves* [Lie06]. Let us recall its definition.

First, fix a flat and proper morphism of schemes $\pi : X \rightarrow S$.

A.1 Definition. An object $E \in D(\mathcal{O}_X)$ is (relatively over S) *perfect and universally gluable* if the following conditions hold.

- There exists an open cover $\{U_i\}$ of X such that $E|_{U_i}$ is quasi-isomorphic to a bounded complex of quasi-coherent sheaves flat over S .
- For any S -scheme $u : T \rightarrow S$ we have

$$R\pi_{T,*} R\mathbf{Hom}_{X_T}(Lu_X^* E, Lu_X^* E) \in D^{\geq 0}(\mathcal{O}_T)$$

where π_T and u_X denote the maps induced by π and u respectively on the base-change X_T .

We denote the category of perfect and universally gluable sheaves on X (over S) as $D_{\text{pug}}(\mathcal{O}_X)$.

If in the definition we take S to be affine and assume $T = S$, then it's clear that gluability has to do with the vanishing of negative self-exts of E . This condition is necessary to avoid having to enter the realm of higher stacks.

A prestack¹⁴ \mathfrak{Mum}_X is defined by associating with an S -scheme $T \rightarrow S$ (the associated groupoid of) the category $D_{\text{pug}}(\mathcal{O}_{X_T})$ of perfect and universally gluable complexes (relatively over T). The restriction functors are defined by derived pullback.

A.2 Theorem (Lieblich). The prestack \mathfrak{Mum}_X is an Artin stack, locally of finite presentation over S .

From now on we fix $\pi : X \rightarrow S$ flat and projective with S a noetherian scheme. We assume all rings and schemes to be locally of finite type over S .¹⁵

We want to construct various open substacks of \mathfrak{Mum}_X , namely stacks of complexes satisfying additional properties. For example we would like to construct the stack of complexes with cohomology concentrated in degrees less or equal than a fixed integer n . The correct way to proceed is by imposing conditions fibrewise on restrictions to geometric points. Let us illustrate a general recipe first. The following diagram comes in handy.

$$\begin{array}{ccccc} X_t & \xrightarrow{t_X} & X_T & \xrightarrow{u_X} & X \\ \downarrow \pi_t & & \downarrow \pi_T & & \downarrow \pi \\ \text{Spec } k & \xrightarrow{t} & T & \xrightarrow{u} & S \end{array}$$

Here T is the base space for our family of complexes, together with its structure map to S , and $t \in T$ is a geometric point. Obviously if we fix a fibre product $X_k = X \otimes_S k$ then all the fibres X_t are canonically identified with X_k . Given a property P , we might define the stack of complexes satisfying P as follows.

$$\mathfrak{Mum}_X^P(T) = \left\{ E \in \mathfrak{Mum}_X(T) \mid \forall \text{ geometric } t \in T, E|_{X_t}^L \text{ satisfies } P \right\}$$

We recall that by $E|_{X_t}^L$ we mean $Lt_X^* E$.

To construct the substacks of \mathfrak{Mum}_X we are interested in we make use of the following lemma.

¹³The author would like to thank Fabio Tonini for patiently explaining to him many things about stacks.

¹⁴We are using the term *prestack* in analogy with term *presheaf*.

¹⁵For what follows, this assumption isn't substantial (as \mathfrak{Mum}_X is locally of finite type over S) but it enables us to use the local criterion of flatness directly.

A.3 Lemma. Let $T \rightarrow S$ be an S -scheme, let $t : \text{Spec } k \rightarrow T$ be a point of T and let $E \in D^b(\mathcal{O}_{X_T})$ be a bounded complex of \mathcal{O}_{X_T} -modules flat over T . Let $n \in \mathbb{Z}$ be an integer. The following statements hold.

(1) $E|_{X_t}^L \in D^{\leq n}(\mathcal{O}_{X_t}) \iff X_t \subset U_{>}$, where

$$U_{>} = \bigcap_{q > n} X_T \setminus \text{supp } H^q(E).$$

(2) $E|_{X_t}^L \in D^{[n]}(\mathcal{O}_{X_t}) \iff X_t \subset U$, where¹⁶

$$\begin{aligned} U &= U_{>} \cap U_f \cap U_{<} \\ U_{>} &= \bigcap_{q > n} X_T \setminus \text{supp } H^q(E) \\ U_f &= \left\{ x \in X_T \mid H^n(E)_x \text{ is a flat } \mathcal{O}_{T, \pi_T(x)}\text{-module} \right\} \\ U_{<} &= \bigcap_{q < n} X_T \setminus \text{supp } H^q(E). \end{aligned}$$

(3) $E|_{X_t}^L \in D^{\geq n}(\mathcal{O}_{X_t}) \iff F \in D^{[n]}(\mathcal{O}_{X_t})$, where $F = \sigma_{\leq n} E$ is the stupid truncation of E in degrees less or equal than n .

$$F^p = \begin{cases} E^p, & \text{if } p \leq n \\ 0, & \text{if } p > n \end{cases}$$

Proof: PROOF OF 1. Let t_X be the inclusion of the fibre $X_t \rightarrow X_T$. As t_X is an affine map we do not lose information on the cohomologies of $E|_{X_t}^L$ after pushing forward back into X_T . We also have isomorphisms

$$t_{X,*} E|_{X_t}^L \simeq E \otimes_{\mathcal{O}_{X_T}}^L t_{X,*} \mathcal{O}_{X_t} \simeq E \otimes_{\mathcal{O}_{X_T}}^L \pi_T^* t_* k$$

where the first follows from the projection formula and the second from base change compatibility. As we are interested in the vanishing of $H^q(E|_{X_t}^L)$ we may restrict to the stalk at a point $x \in X_t$. Taking stalks at x gives us isomorphisms

$$(A.4) \quad H^q(E|_{X_t}^L)_x \simeq H^q\left(E_x \otimes_{\mathcal{O}_{T,t}}^L k\right).$$

We have the page two spectral sequence of the pullback

$$(A.5) \quad L^p t_X^* H^q(E) \implies H^{p+q}(E|_{X_t}^L).$$

which, at a point $x \in X_t$ and using the isomorphism (A.4), boils down to

$$(A.6) \quad \text{Tor}_{-p}^{\mathcal{O}_{X_t}}(H^q(E)_x, k) \implies H^{p+q}(E|_{X_t}^L)_x.$$

Let now q be the largest integer such that $H^q(E) \neq 0$. From the spectral sequence (A.6) we have

$$H^q(E|_{X_t}^L)_x \simeq H^q(E)_x \otimes_{\mathcal{O}_{T,t}} k.$$

Hence, by Nakayama, $H^q(E|_{X_t}^L)_x = 0$ if and only if $x \in X_T \setminus \text{supp } H^q(E)$ and finally

$$H^q(E|_{X_t}^L) = 0 \iff X_t \subset X_T \setminus \text{supp } H^q(E).$$

PROOF OF 2. Using 1. we can assume that $E|_{X_t}^L \in D^{\leq n}(\mathcal{O}_{X_t})$. By the spectral sequence (A.5) we have that $H^{n-1}(E|_{X_t}^L) \simeq L_1 t_X^* H^n(E)$. Again, we may pass on to the stalk at a point $x \in X_t$ and (A.6) yields

$$H^{n-1}(E|_{X_t}^L)_x \simeq \text{Tor}_1^{\mathcal{O}_{X_t}}(E_x, k)$$

the vanishing of which is equivalent, by the local criterion for flatness, to $H^q(E)_x$ being a flat \mathcal{O}_{X_t} -module.

We can thus assume that $X_t \subset U_{>} \cap U_f$. Once more, from the spectral sequence (A.6) we have that $H^{n-1}(E|_{X_t}^L) \simeq t_X^* H^{n-1}(E)$ and we proceed as in the proof of 1.

PROOF OF 3. Consider the page one spectral sequence

$$L^q t_X^* E^p \implies H^{p+q}(E|_{X_t}^L)$$

¹⁶The superscript $[n]$ is a shorthand for the awkward $[n, n]$ and stands for ‘concentrated in degree n ’.

from which we get isomorphisms

$$H^p(E|_{X_t}^L) \simeq H^p(t_X^* E)$$

as a consequence of flatness of the E^q 's. Thus, for $p < n$,

$$H^p(E|_{X_t}^L) = 0 \iff H^p(t_X^* E) = 0 \iff H^p(t_X^* F) = 0.$$

■

A.7 Proposition. Define the prestack $\mathfrak{Mum}_X^{\leq n} = \mathfrak{Mum}_X^{[-\infty, n]}$ by assigning to each S-scheme T the groupoid

$$\mathfrak{Mum}_X^{\leq n}(T) = \{E \in \mathfrak{Mum}_X(T) \mid \forall \text{ geometric } t \in T, E|_{X_t}^L \in D^{\leq n}(\mathcal{O}_{X_t})\}$$

with restriction functors induced by \mathfrak{Mum}_X . The prestack $\mathfrak{Mum}_X^{\leq n}$ is an open substack of \mathfrak{Mum}_X .

Proof: That $\mathfrak{Mum}_X^{\leq n}$ satisfies descent is a direct consequence of descent for \mathfrak{Mum}_X . To prove that it is indeed an open substack it is sufficient to prove that for any affine S-scheme T, together with a morphism $T \rightarrow \mathfrak{Mum}_X$ corresponding to a complex $E \in \mathfrak{Mum}_X(T)$, the set

$$V = \{t \in T \mid E|_{X_t}^L \in D^{\leq n}(X_t)\}$$

is an open subset of T.

By Lemma A.3 1. we know that $t \in V$ if and only if $X_t \subset U_{>}$ (notice that by our assumptions the complex E is bounded). Thus $\pi_T(X_T \setminus U_{>}) = \pi_T(X_T) \setminus V$. The set $U_{>}$ is open as the sheaves $H^q(E)$ are quasi-coherent and of finite type. Finally, the sets $\pi_T(X_T)$ and $\pi_T(X_T \setminus U_{>})$ are closed, being the image of closed subsets under a proper map. Thus, V is open. ■

Notice that the condition of being concentrated in degrees less or equal than n is in fact a global condition, i.e. we could have requested $E \in D^{\leq n}(\mathcal{O}_{X_T})$ directly.

We now impose on our complexes the further condition of being concentrated in a fixed degree $n \in \mathbb{Z}$. This stack will be isomorphic to the stack of coherent sheaves shifted by $-n$.

A.8 Proposition. Define the prestack $\mathfrak{Mum}_X^{[n]}$ by assigning to each S-scheme T the groupoid

$$\mathfrak{Mum}_X^{[n]}(T) = \{E \in \mathfrak{Mum}_X^{\leq n}(T) \mid \forall t \in T, E|_{X_t}^L \in D^{[n]}(\mathcal{O}_{X_t})\}$$

with restriction functors induced by \mathfrak{Mum}_X . The prestack $\mathfrak{Mum}_X^{[n]}$ is an open substack of $\mathfrak{Mum}_X^{\leq n}$.

Proof: The proof follows along the lines as the previous one. It suffices to show that for any affine scheme T, together with a map $T \rightarrow \mathfrak{Mum}_X^{\leq n}$ corresponding to a complex $E \in \mathfrak{Mum}_X^{\leq n}(T)$, the set

$$V = \{t \in T \mid E|_{X_t}^L \in D^{[n]}(\mathcal{O}_{X_t})\}$$

is an open subset of T. By Lemma A.3 2. we know that $t \in V$ if and only if $X_t \subset U$. The sets $U_{<}, U_{>}$ are open as the sheaves $H^q(E)$ are quasi-coherent and of finite type. The set U_f is open by the open nature of flatness [EGAIV-3, Théorème 11.3.1]. Thus U is open and we conclude as in the previous proof. ■

When $n = 0$ we get back the ordinary stack of coherent sheaves on X .

We now turn to the opposite condition: being concentrated in degrees greater or equal than a fixed $n \in \mathbb{Z}$.

A.9 Proposition. Define the prestack $\mathfrak{Mum}_X^{\geq n} = \mathfrak{Mum}_X^{[n, \infty]}$ by assigning to each S-scheme T the groupoid

$$\mathfrak{Mum}_X^{\geq n}(T) = \{E \in \mathfrak{Mum}_X(T) \mid \forall t \in T, E|_{X_t}^L \in D^{\geq n}(\mathcal{O}_{X_t})\}$$

with restriction functors induced by \mathfrak{Mum}_X . The prestack $\mathfrak{Mum}_X^{\geq n}$ is an open substack of \mathfrak{Mum}_X .

Proof: As in the previous proofs we consider a complex $E \in \mathfrak{Mum}_X(T)$ corresponding to a morphism $T \rightarrow \mathfrak{Mum}_X$ and prove that the set

$$V = \{t \in T \mid E|_{X_t}^L \in D^{\geq n}(X_t)\}$$

is an open subset of T. By Lemma A.3 3. this set is equal to

$$V = \{t \in T \mid F|_{X_t}^L \in D^{[n]}(X_t)\}$$

which is open by the previous proof. ■

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